SOME TOPOLOGICAL EXCITATIONS IN PHYSICS

An Honors Thesis in Physics

submitted by

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Introduction

This paper is an exposition of some of the ideas that I have been pursuing as part of the honors program in physics at Stanford university. It concerns some very recent (all within the last ten years) ideas that apply topological concepts to various areas in physics. Though a number of the ideas and derivations within are my own, my major contribution has been to bring together in one place ideas that are scattered all over the literature. I have tried to present things in a self-contained fashion (with the exception of differential forms which were useful in examples at several points but not central enough to warrant a full exposition). This has been a very useful exercise for solidifying my understanding of the various topics.

In the first section I discuss the structure of physical theory and the possible role of topology, using the various forms of Maxwell's equations as an example. The second section gives some of the background topological ideas used later. The third section discusses the intuitively appealing topological excitations that appear in one-dimensional field theories and shows why they don't straightforwardly appear in higher dimensional theories. Therefore the fourth section develops the general structure of gauge theories within which they can appear. The fifth section demonstrates a number of topological objects that can appear in gauge field theories. The last section shows a similar phenomenon in

statistical mechanics. First various features of the Ising model are discussed. The role of vortices with non-trivial topological winding number in the phase transitions of the X-Y model and a variety of other systems then concludes this discussion.

I would like to thank Dr. Leonard Susskind for being my adviser and for teaching me a lot about physics.

I. Why might topology be relevant to physics?

How is physics possible? A priori one might expect that one would need at least one mark on the paper for every event in the world that one is attempting to describe. In such a situation the world would be easier to comprehend than any description. The way that physics gets out of this tangle is through the observation that there are aspects of the world which are the same. It is possible to describe the similarity between events in much less space than a description of all the events involved. Through many applications of this process one arrives at its quintessential version in the concept of universal physical law and the enterprise of physics is on.

From such a perspective, the task of physics is to find and describe the similarities in the structure of the universe. What is the appropriate language for such a description? First attempts might be to try to symbolize identical events . Such attempts are bound to fail for, by definition, an exact total symmetry is not observable. If two events don't differ in any aspect they are indistinguishable. What is needed is a more subtle description which can capture, for example, the similarity of two events at different places or the similarity only in shape of a red and a blue marble.

The key is to shift attention from the events which are similar to the set of all transformations under which the symmetry involved is invariant. It is a remarkable fact that one may abstract a small number of common features of these

sets which serve to characterize them exactly. Objects with these features had arisen in the theory of equations and mathematicians called them groups. Some of their properties were known to Lagrange in 1800, but the first abstract definition was given in 1849 by Arthur Cayley[1]. A group is a set G with a "multiplication" defined so that the product of two elements of G is again an element of G and the following axioms are satisfied:

1. If $a,b,c \in G$ then $a(bc)=(ab)c$
2. There is an identity element $e \in G$ such that for any $a \in G$ it is true that: $ae=ea=a$
3. Given any $a \in G$ there exists an element $a^{-1} \in G$ so $a^{-1}a=aa^{-1}=e$

One can easily see that symmetry transformations satisfy these axioms (with doing nothing as identity and undoing a transformation as its inverse). Further, most types of groups show up in physics so the characterization is a good one. From this simple beginning has developed an enormous mathematical subject. A lot of the concepts developed have direct relevance to the physical situation (e.g. subgroups, quotient groups, product groups, and homomorphisms which are maps between groups which preserve the product). Some examples of symmetry groups are the Poincare' group, the Galilean group, their common subgroup: the rotation group, isospin rotation and various other particle symmetries, crystallographic point groups and permutation groups of identical particles. We'll see other uses for groups shortly.

Symmetry arguments give qualitative information (which is what we're ultimately after anyway) like selection rules and possible quantum numbers. They are therefore at the high-level end of the spectrum of levels of description. When we wish to

talk about individual objects or examples it is convenient to break the symmetry of the high-level description. Thus we introduce coordinate systems, label identical particles, work in particular gauges, reference frames, and representations, and introduce explicit generators for our groups, bases for our vector spaces and phase conventions all over. We then add the requirement that nothing physical can depend on our choice. Description on this level is often mechanically easy to deal with and is crucial for explicit calculations. Such conveniences are irrelevant to the physics, however. As the systems get more complex such descriptions get increasingly unwieldy and more and more of the formalism becomes irrelevant to the physical situation at hand.

It is for this reason that some powerful methods for "coordinate-free" high-level description have been developed. We find that when described in a coordinate free manner, physics looks simple. Often it is considerably easier to see what's going on at this level and more high-level concepts are expressible. A few examples are in order: In quantum mechanics the Heisenberg version of matrix mechanics or the Schrödinger wave mechanics are well-suited for specific problems but necessitate a large number of special cases. These get unified in the Dirac bra and ket notation. Another example is special relativity which has lots of complicated special cases which unify when things are expressed as 4-vectors in spacetime.

For an explicit example, we look at the development of

the vacuum Maxwell's equations. Maxwell's original version consisted of 8 partial differential equations and 6 more to define the potential:

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=4\pi\rho \qquad \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0$$

$$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}-\frac{1}{c}\frac{\partial E_x}{\partial t}=\frac{4\pi}{c}J_x \qquad \frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}-\frac{1}{c}\frac{\partial E_y}{\partial t}=\frac{4\pi}{c}J_y$$

$$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}-\frac{1}{c}\frac{\partial E_z}{\partial t}=\frac{4\pi}{c}J_z \qquad \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{1}{c}\frac{\partial B_x}{\partial t}=0$$

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}+\frac{1}{c}\frac{\partial B_y}{\partial t}=0 \qquad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{1}{c}\frac{\partial B_z}{\partial t}=0$$

$$\frac{\partial \phi}{\partial x}=-E_x \qquad \frac{\partial \phi}{\partial y}=-E_y \qquad \frac{\partial \phi}{\partial z}=-E_z \qquad \frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}=B_x$$

$$\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}=B_y \qquad \frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=B_z$$

This was the form Einstein used in the 1905 special relativity paper.[2] In this form the relativistic transformation properties are an un-intuitive mess.

Next came the spatial coordinate-free version which is now standard:

$$\vec{\nabla}\cdot\vec{E}=4\pi\rho \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{B}-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}=\frac{4\pi}{c}\vec{J} \qquad \vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0$$

$$\vec{\nabla}\Phi=-\vec{E} \qquad \vec{\nabla}\times\vec{A}=\vec{B}$$

Symbols like $\vec{\nabla}$ and $\vec{E}$ and the concept of div, grad, and curl

eliminate arbitrary coordinates and provide us with new higher-level concepts. Vector analysis was still treated with caution by some in the 1940's![3]

Next came the fully relativistic tensor notation:

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \qquad \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0 \qquad \frac{\partial F^{\mu\nu}}{\partial x^\nu} = J^\mu$$

Here the electro-magnetic unification becomes apparent as does the relativistic invariance. But we still have particular indices and coordinate transformation matrices etc. cluttering up the concepts.

In the 1920's and 1930's Elie Cartan and his coworkers developed coordinate-free notation for differential geometry. This finally came to be used in theoretical physics about 15 years ago. In this notation Maxwell's equations are:[4]

$$F = dA \qquad dF = 0 \qquad d^*F = {}^*J$$

Here the A and J are one-forms (covariant vector fields) and F is a two-form (covariant anti-symmetric second rank tensor field). The d operator is a beautiful example of a high-level notion unifying lots of seemingly disparate lower levels. It is the dual notion (talks about functions of the object it is dual to) to the boundary operator ∂ (give it a manifold and it returns the boundary).[5] An n dimensional manifold and an n-form contract via integration to give a number. The duality is manifest through the generalization of Newton's, Gauss's, Green's and Stoke's Theorems: $\int_{\partial C} w = \int_C dw$. The d operator generalizes

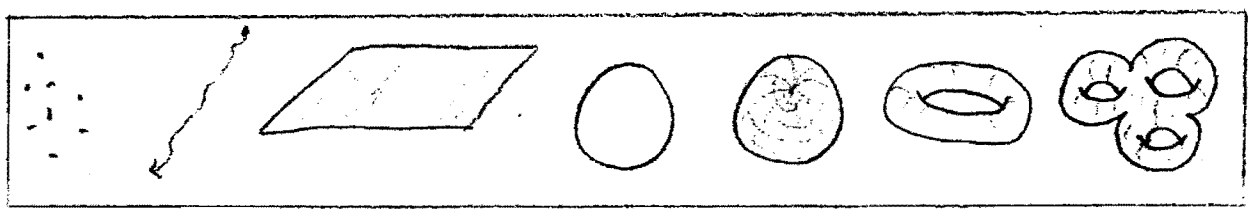
the gradient (0-forms to 1-forms), the curl (1-forms to 2-forms), and the divergence (2-forms to 3-forms) in three dimensions. From the general concept: the boundary of a boundary is zero (e.g. the 2-dimensional surface of a 3-dimensional solid ball has no 1-dimensional edges) we get the general concept $dd=0$. In three dimensions this subsumes: curl grad=0 and div curl=0. In the case of E&M the first Maxwell equation follows from this purely geometric reason: $ddA=dF=0$. Similarly if we let d act on the second Maxwell equation we obtain: $dd*F=d*J=0$ which is just the continuity equation for the charge-current 3-form.

This last example is a beautiful example of a topological idea (these structures are the base for homology and cohomology theory) shedding light on the physical structures involved. The idea is that in our conventional formulations we miss a kind of symmetry by imposing too much structure. There are a whole class of physically relevant properties where continuity and not distance is the crucial factor. These properties are the domain of topology.

II. What is topology?

"Topology is the study of topological spaces and continuous functions between them."[6] A topological space is a set of points with a notion of closeness (a specification of the open subsets) but, in general, no notion of distance. A large area, called point-set topology, is devoted to the study of properties at this level. As of now, most of this work is irrelevant for physics and we need only consider the very "nicest" of topological spaces. In fact, we will deal only with manifolds, whose local properties are like Euclidean space (there is an open set about each point in the manifold that is homeomorphic to an open ball in R^n and in addition the manifold inherits the local differentiable structure). We thus look at smoothness and continuity but not distance.

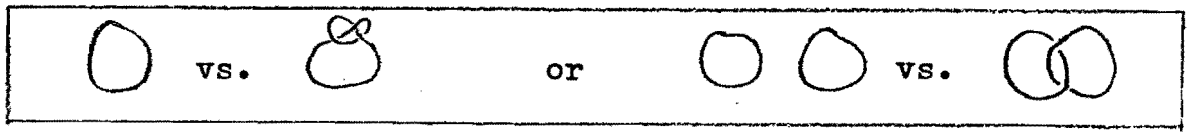
Some typical examples of manifolds are: Euclidean spaces (of any dimension, e.g. points, lines, planes, etc.), spheres, open balls (solid balls without the bounding sphere), toruses (with any number of holes), projective planes (solid balls with diametrically opposite points identified), etc. These objects are the elementary particles of topology from which more complicated things are built.



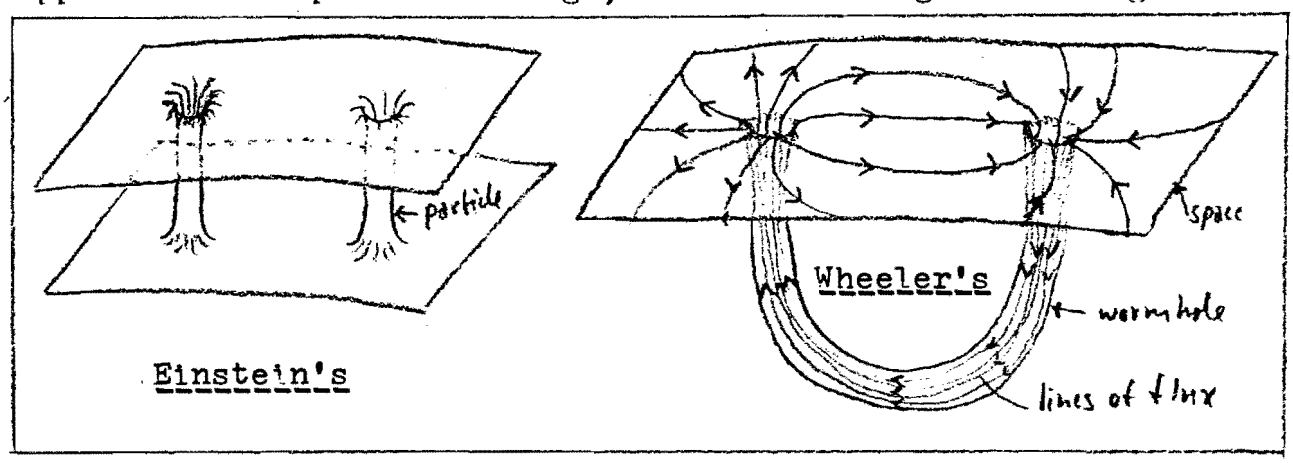
The area of topology which we will be concerned with is the more combinatorial algebraic topology. This area had its beginnings around 1890 with Poincaré, who invented a lot of

the concepts in order to resolve questions of stability in celestial mechanics and mechanics in general.[7] Algebraic topology really began as an "experimental" subject, trying to understand already existing "natural" sorts of objects.

A typical problem is to characterize properties of topological objects which are invariant under continuous distortions (bending and stretching but no ripping or glueing). Thus, for example, the "knottedness" or "linkedness" of loops of string is independent of how you bend them around (for technical reasons one must actually consider the topology of the space outside the string in this particular case).

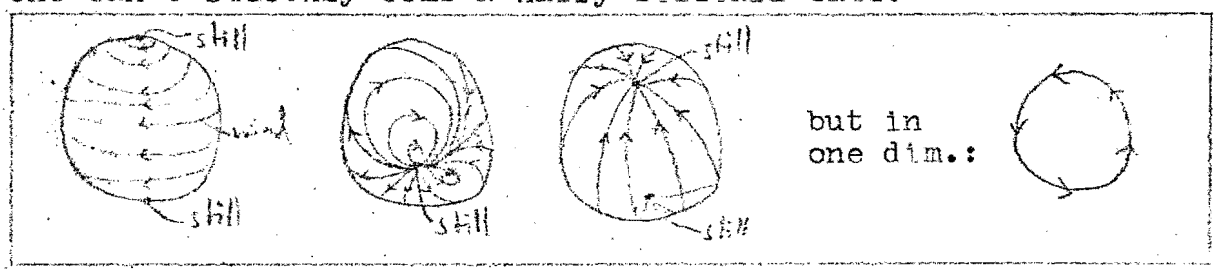


This sort of idea has had a powerful effect on physicists in the past since it is such an intuitively appealing way of getting conserved objects in discrete packets. Einstein wanted to model elementary particles as little "tunnels" between two planes. Wheeler wanted to view electric charges using "wormholes" connecting distant points in space, within which were trapped lines of electric flux. One opening would appear to be a positive charge, the other a negative charge.[8]

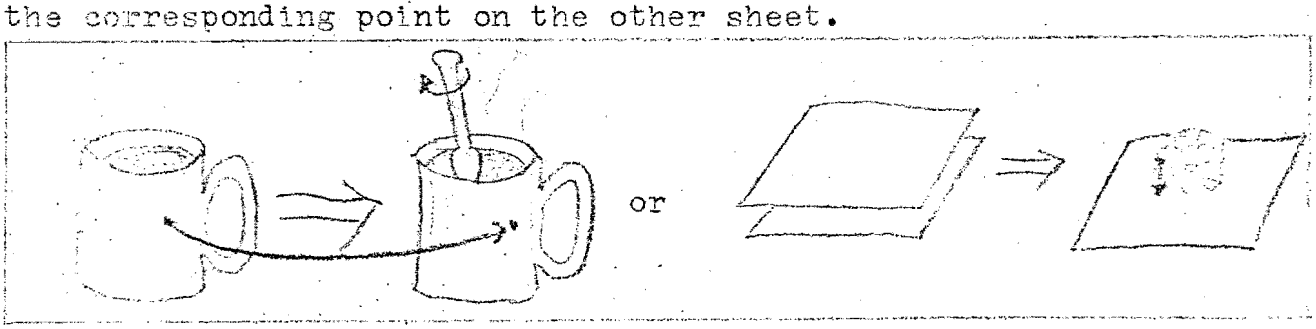


While these two ideas were never refined into complete models, we'll see some similar things popping up in more modern formulations.

To get some of the flavor of ideas in algebraic topology, we'll look briefly at two classical theorems of Brouwer (1911)[9]. The first states that there are no continuous non-vanishing tangent vector fields on even dimensional spheres. Thus the wind must be still at some point on the earth's surface and one can't smoothly comb a hairy billiard ball.



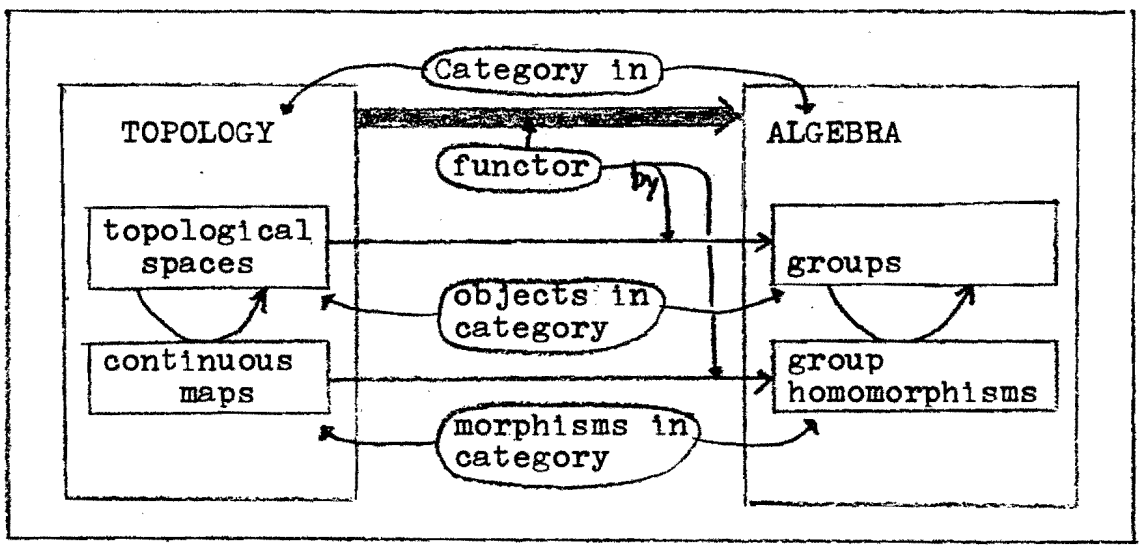
The other is the so-called fixed-point theorem. This says that for $n \geq 0$, every continuous map from the n-dimensional ball to itself must leave at least one point fixed. Thus if one stirs a cup of coffee (no splashing allowed), at least one point of the fluid will end up at its starting point. Or if you take two identical sheets of paper and crumple one up and set it on the other, at least one point must be directly over the corresponding point on the other sheet.



A typical use of the fixed point theorem is to show that there must be a solution to some nonlinear differential equation.

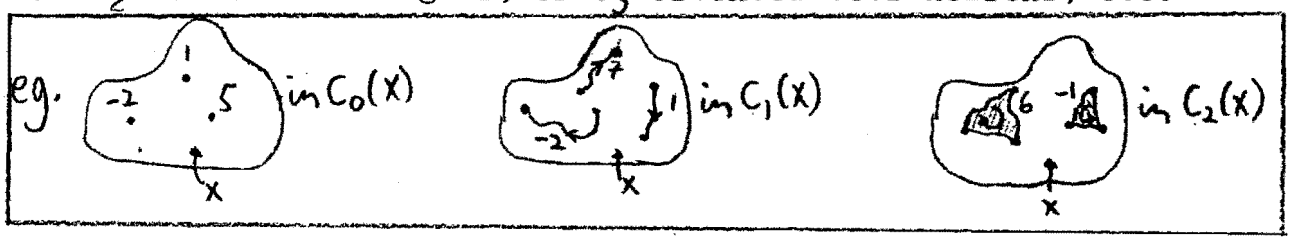
The idea is to write the equation as an equality between the desired function and some continuous transformation thereof. Thus we map continuously a region of function space to itself and there must be a fixed point which is the desired solution.

These theorems appear to be so general that one wonders how one would even begin to prove them. The key idea is contained in an area of mathematics known as category theory.[10] One defines a collection of objects and morphisms between them as a category. One then defines maps between categories called functors that associate structures from different parts of mathematics in a well defined way. In our case, topological structures get mapped into algebraic ones. Amazingly, complicated problems on the topology side sometimes turn into tractable algebraic problems. The structure looks like:

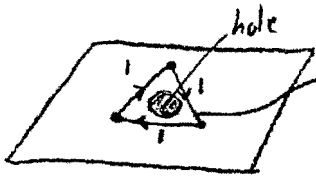


The main two examples of this map are homology theory and homotopy theory. We have already seen one example of homology (cohomology) theory ideas applied to differential

forms. In homology theory, given a topological space X, one defines a group $C_n(X)$ of finite chains for each dimension n. The elements of $C_n(X)$ are linear combinations with integer coefficients of maps from n-dimensional oriented tetrahedra into X, such that all but a finite number of the coefficients are zero. Thus the elements of C_0 are associations of integers with points in the space; of C_1 integers with directed paths; of C_2 oriented triangles; of C_3 oriented tetrahedrons; etc.



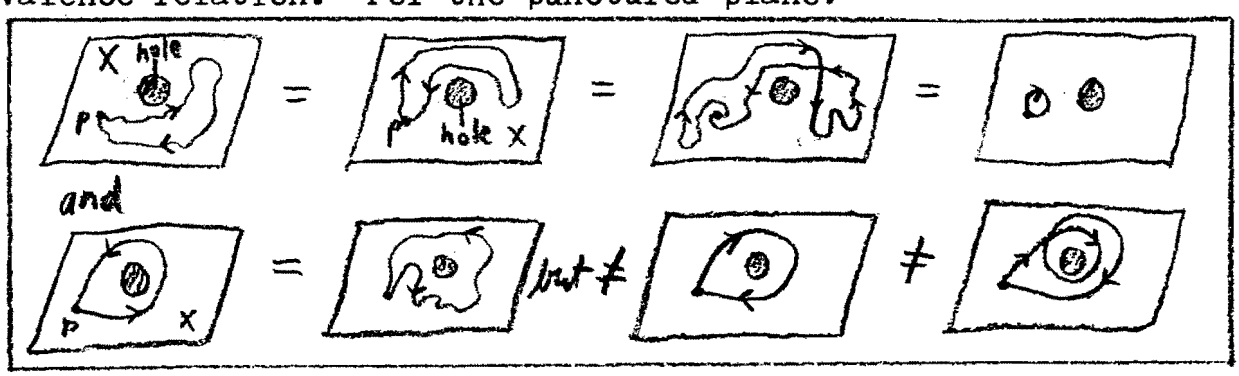
The group operation is pointwise (pathwise, trianglewise, tetrahedronwise, etc.) addition of the integers with the convention that opposite orientation changes the sign (so $\overset{1}{\bullet}\!\!\rightarrow = -\overset{1}{\bullet}\!\!\rightarrow$ or $5\,\triangle = -5\,\triangle$). One then defines a homomorphism $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ which takes any element to its oriented boundary (so $5\,\overset{\partial}{\longmapsto}\, 5\bullet - 5\bullet$ and $6\,\triangle \overset{\partial}{\longmapsto} \triangle + \triangle + \triangle$). The image of ∂ is a subgroup of $C_{n-1}(X)$ called B_{n-1} and its elements are the boundaries of n-dimensional chains. The kernel (elements that get taken to zero) of ∂ is a subgroup of $C_n(X)$ called Z_n and its elements are those chains that have no boundary. We saw that the boundary of a boundary is zero so Z_n is a subgroup of B_n. If the space is contractible to a point, then all boundaryless chains are the boundary of something in the space. If, however, there are "holes" then chains of the appropriate dimension can go around the hole in such a way that they have no boundary, yet are not the boundary

of anything. Eg.: 

element of C_1 with no boundary (since endpoints cancel) yet no triangle chain has it as its boundary (because of hole)

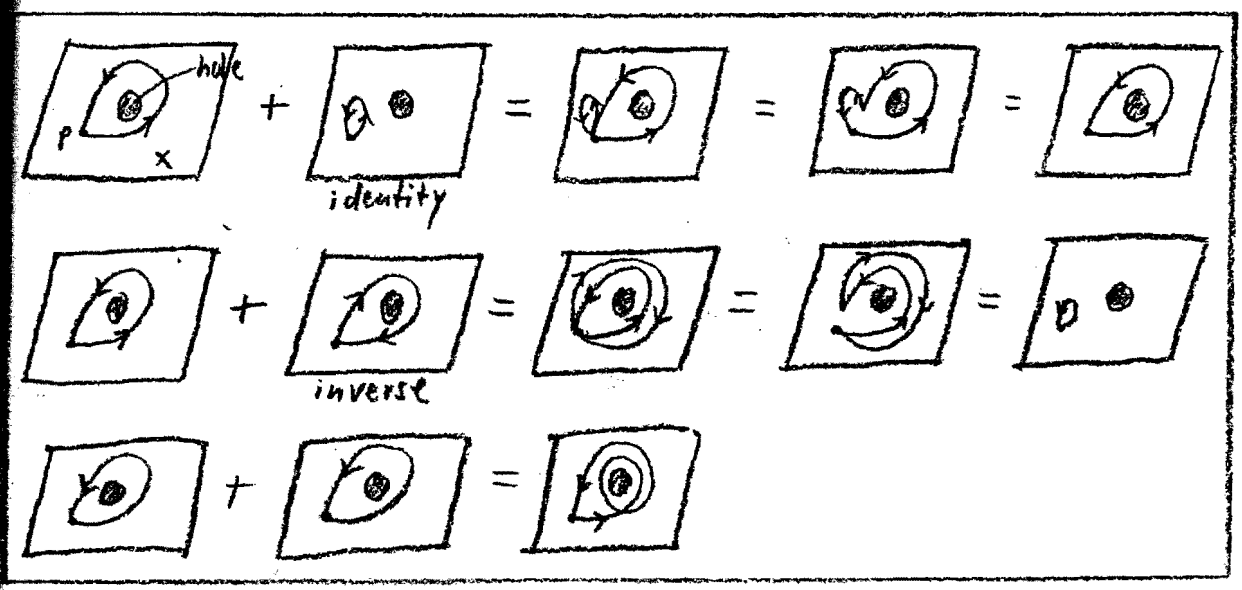
Thus we are motivated to define the nth homology group: $H_n=B_n/Z_n$. It is a measure of the kinds of "holes" in the space. The sequence of groups H_n is the algebraic object we associate with the topological space X. II. All of these notions carry over to differential forms where they generalize the notions of exact and closed, familiar from the theory of ordinary differential equations.

The homotopy group map is less powerful but easier to visualize. We begin by discussing the first homotopy group, also known as the fundamental group, of our topological space X. We consider maps from the oriented circle S^1 (in general, the n-dimensional sphere S^n is the set of points in R^{n+1} satisfying $\sum_4 (x^4)^2=1$) into X such that the image includes a given point $p \in X$. We say that two such maps are homotopic if they can be continuously distorted into one another (these notions may easily be made precise at the expense of cluttering up the notation). We can easily see that this is an equivalence relation. For the punctured plane:



The elements of the first homotopy group will be these equivalence classes. The product will be the class of the path

formed by first going along a representative of the first class and following it with a representative of the class we're taking the product with. The identity is the class of loops contractible to a point. The inverse of a loop is the class of the same loop taken as going in the other direction. The resulting group structure is denoted by $\pi_1(X)$.[12] With X as before:



We can see that the equivalence classes may be labeled by the number of times we loop around the hole (positive for counter-clockwise, negative for clockwise, for instance). The class addition is then just like addition of integers (go around twice plus go around three times equals go around five times, etc.). The plane with a hole in it is topologically equivalent to the circle S^1, so we have shown $\pi_1(S^1)=Z$ (Z is the group of integers under addition).

Consider the case where X is the two dimensional torus:  . We can label the class by the number of times we loop around the center hole  and the number of times we loop through it  . We can see that these numbers add independently

so the group is $Z \times Z$ (the direct product of two groups $A \times B$ is the group of ordered pairs (a,b), $a \in A$, $b \in B$, with multiplication defined by: $(a,b)(a',b')=(aa',bb')$). This is a special case of the more general theorem $\pi_n(A \times B)=\pi_n(A) \times \pi_n(B)$. (The torus is $S^1 \times S^1$.)

As another example consider the projective plane  (disc with opposite points on the "edge" identified). We have two kinds of paths, those that cut an odd number of times through the "edge" like  or  and those that cut across an even number of times like  or  . The sum of two odd paths is even so the group is Z_2 ($=Z/2Z$ the group containing elements 0,1 so that: $0+0=0$, $1+0=0+1=1$, $1+1=0$).

The above notions may be generalized to the nth homotopy group of a space written $\pi_n(X)$. Everything is the same except that instead of considering maps from S^1, we consider maps from S^n. It's not too hard to see that $\pi_n(S^m)=0$ for $n<m$ (you can't lasso a basketball) and that $\pi_n(S^n)=Z$ (sphere can go once around itself, twice, etc.). Surprisingly enough, it is not the case that $\pi_n(S^m)=0$ for $m<n$ in general. This is shown by the Hopf map: $\pi_3(S^2)=Z$. (The unit quaternions $a+bi+cj+dk$, $a^2+b^2+c^2+d^2=1$ are an example of S^3. Multiply by $e^{i\theta}$ and you can mix a and b and mix c and d but not between them. Thus if we identify all those points related by multiplication by $e^{i\theta}$ we get two parameters and thus S^2 which can now map non-trivially onto S^2.)[13]

Now that we have considered some algebraic topology, we must see how it relates to some topological algebra. A continuous

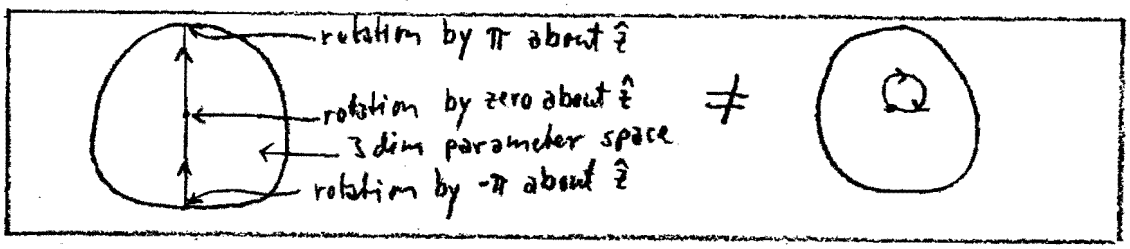
group is a group that is, in addition, a topological space and the group product and inverse are continuous functions. One example is the additive group of real numbers with the usual topology. Another important such group is U(1), the group of complex numbers $e^{i\theta}$, θ real, under multiplication with the topology induced from the complex plane. This space is isomorphic to S^1. Almost all of the interesting topological aspects of physics arise from the non-trivial topological structure of the groups involved. Thus a lot of the richness of quantum mechanics comes from the use of complex numbers, which have a multiplicative structure: $U(1)\times(R-\{0\})$. Thus the distinct L_z eigenvalues in the hydrogen atom can be viewed as arising from homotopically distinct maps from the S^1 determined by the polar angle ϕ to the S^1 of the U(1) phase of the wavefunction.

Most of the continuous groups in physics are in addition Lie groups (there is a parameter space and group operations are analytic functions of the parameters) and in particular usually one of the classical groups. These are subgroups of the groups GL(N) of non-zero determinant N×N matrices under multiplication.[14]

An important example is the 2-dimensional, unit-determinant, unitary matrices SU(2). An element $U=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the conditions det U=ad-bc=1 and $U^\dagger=U^{-1}=\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ so that $d^*=a$ and $c^*=-b$ implying $|a|^2+|b|^2=1$. So each matrix of SU(2) is associated with a point on the three dimensional sphere S^3

in 4-dimensional space (a,b complex). Thus the topology of SU(2) is isomorphic to S^3.[15]

Another important group is that of rotations in 3-dimensions SO(3). We define a 3-dimensional parameter space by identifying the direction of a line from the origin to the point in parameter space as the axis about which we rotate and the distance from the origin to the point as the angle about which to rotate about this axis. We must identify the points π and $-\pi$ along this line since they correspond to the same rotation. Thus the topology is that of the 3-dimensional projective plane (3-dimensional solid ball with diametrically opposed points on the surface identified). But we saw above that $\pi_1(P^3)=Z_2$. Thus there are two classes of rotations which cannot be continuously deformed into one another.



Rotation by 2π about an axis is in the other class from no rotation at all. It is for this reason that we get double-valued representations of the rotation group and thus spinorial quantities with half-odd integer angular momenta.[16]

It turns out that $SU(2)/Z_2$ is isomorphic as a group and as a topological space to SO(3). The above is a particular case of the result $\pi_1(SU(N)/Z_N)=Z_N$ for $N\geq 2$. The embedding of Z_N in SU(N) is as $e^{ia2\pi/N}I$, $a=0,1,\ldots,N-1$. This result comes

about because a closed path in $SU(N)/Z_N$ must have come from a path in SU(N) beginning at the identity I but ending on any of the N elements of Z_N.that get mapped to the identity. These paths are clearly inequivalent and have the group structure Z_N.[17]

III. Solitons and the Sine-Gordan equation

We shall now look at the way some of these topolological concepts manifest themselves in physics. The main idea we shall consider is that of topological excitations. These may be thought of as non-zero energy packets trapped by the topology of the situation. We shall see that there are also topological conservation laws and so the intuitive picture of interacting particles lends itself to these excitations.

"Solitary waves" with some of these characteristics were known in classical physics to arise from certain non-linear differential equations. In 1834 Scott-Russell had his "... first chance interview with that singular and beautiful phenomena" while riding on horseback alongside a canal. He observed a water wave from a boat that "accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or dimunition of speed." [18]

In 1895, Korteweg and deVries derived the equation: $\phi_t + \alpha\phi\phi_x + \phi_{xxx} = 0$ (now called the KdV equation) for shallow water waves, ignoring dissipation, and showed that it has solitary wave solutions. These were thought to be due to special initial conditions until Zabusky and Kruskal used a computer model to show that two such waves retained their shape and velocity after a collison (a little time delay

being the only effect). They termed such waves "solitons". Since then this term has come to be applied to any non-dispersive waves (along with the names "kink", "lump", etc.). We will see that in at least some cases their behavior stems from topological properties.

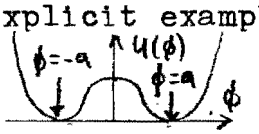
We will consider some classical field equations. The potentially relevant quantum field theoretic versions make the variables into fields of operators on the Hilbert space of state vectors. At present, these equations can only be dealt with perturbatively most of the time. Using Feynman path integrals, it can be shown that the perturbation should be taken about the classical solutions. Thus there is the possibility that topologically distinct classical solutions will identify distinct quantum sectors as well.

Throughout we take $\hbar=c=1$ and choose the metric $g_{tt}=1$, $g_{xx}=-1$, where x represents all spatial dimensions. We begin with a scalar field ϕ in one spatial dimension x. We consider a Lagrangian density: $\mathcal{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-U(\phi)=\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2-\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2-U(\phi)$ where U is a potential energy dependent on ϕ but not on its derivatives. In the cannonical manner we find the Euler-Lagrange equations of motion to be: $\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)=0$ $=-U'(\phi)-\partial_\mu\left(2\cdot\frac{1}{2}\partial^\mu\phi\right)=-U'(\phi)-\partial_\mu\partial^\mu\phi$ or $\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}=-U'(\phi)$ and the Hamiltonian density: $\mathcal{H}=\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+U(\phi)$. Thus the energy is: $H=\int_{-\infty}^{\infty}\left(\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+U(\phi)\right)dx$.

Considering constant ϕ we see that for the energy to be bounded below, $U(\phi)$ must be. We can shift the energy scale so that the minimum of U is defined to be zero energy. Any

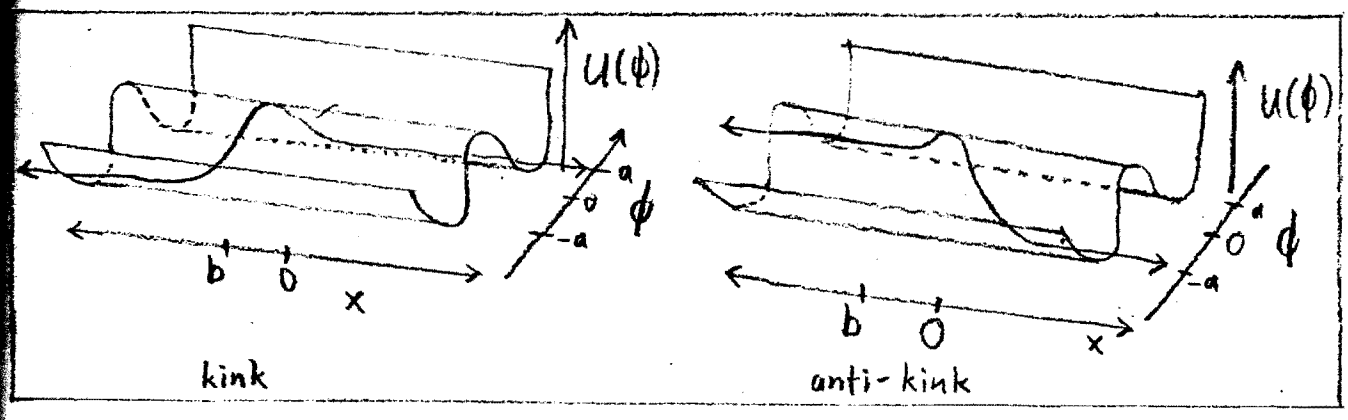
solution with finite energy (i.e. the physical ones) must have ϕ approach one of the zeroes of U as $x \to \pm\infty$ as we can see from the expression for H. If U has only one zero $\phi=\phi_0$ then as $x \to \pm\infty$, ϕ must go to ϕ_0. Such solutions disperse in the usual manner until the energy is spread over large regions of space.

Consider now the possibility that $U(\phi)$ has two zeroes, say $U(\phi_1)=U(\phi_2)=0$. Now we can only say that as $x \to \pm\infty$, ϕ either goes to ϕ_1 or to ϕ_2. If $\phi(\infty) \to \phi_1$ and $\phi(-\infty) \to \phi_1$ or if $\phi(\infty) \to \phi_2$ and $\phi(-\infty) \to \phi_2$ we're in the same dispersive situation as before. If $\phi(\infty) \to \phi_1$ and $\phi(-\infty) \to \phi_2$, however, somewhere between $x=-\infty$ and $+\infty$, ϕ must be different from ϕ_1 and ϕ_2 (it has to cross the higher energy region $\phi_1 < \phi < \phi_2$). This energy can never disperse since it is trapped by the topology of the situation.

As an explicit example consider $U(\phi)=\frac{\lambda}{2}(\phi^2-a^2)^2$ which looks like: 

. We can find a time-independent, localized energy, solution. With $\frac{\partial\phi}{\partial t}=0$ the equation of motion becomes $\frac{\partial^2\phi}{\partial x^2} = U'(\phi) = 2\lambda\phi(\phi^2-a^2)$ so $\frac{\partial^2\phi}{\partial x^2}\frac{\partial\phi}{\partial x} = 2\lambda\phi(\phi^2-a^2)\frac{\partial\phi}{\partial x}$ $= \frac{1}{2}\frac{\partial}{\partial x}\left(\left(\frac{\partial\phi}{\partial x}\right)^2\right) = \frac{\partial}{\partial x}\left(\frac{\lambda}{2}(\phi^2-a^2)^2\right)$ so $\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 = \frac{\lambda}{2}(\phi^2-a^2)^2 + C$ The constant C must be zero since at infinity $U(\phi)$ vanishes and so does $\frac{\partial\phi}{\partial x}$. So $\frac{\partial\phi}{\partial x} = \pm\sqrt{\lambda}(\phi^2-a^2) \Rightarrow \frac{1}{\sqrt{\lambda}}\int_0^{\phi}\frac{d\phi'}{a^2-\phi'^2} = \pm\int_x dx = \pm(x-b)$ So $\frac{1}{2\sqrt{\lambda}a}\log\left(\frac{a+\phi}{a-\phi}\right) = \pm(x-b)$ or $\phi(x) = a\tanh\left(\pm\sqrt{\lambda}\,a(x-b)\right)$. [19]

This solution is remarkable in a number of respects. Let us first draw a sketch to get some intuitive feel for its properties:

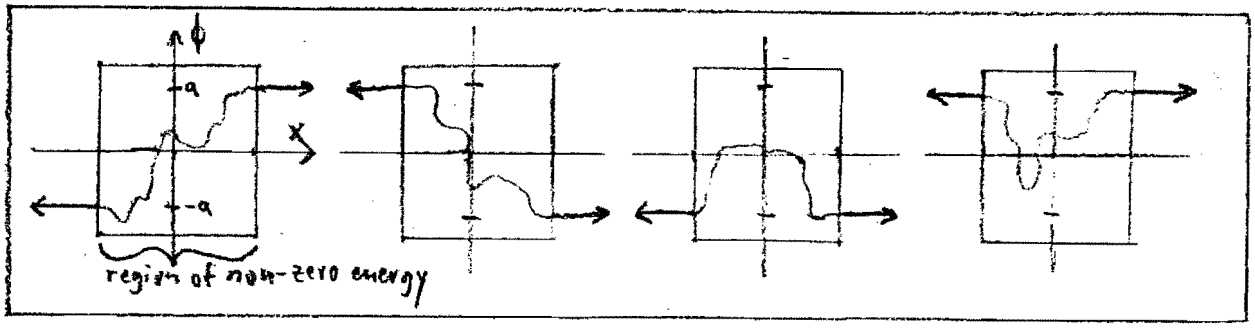


The solution with the plus sign is called a kink and the one with the minus sign an anti-kink. Notice that their shapes are stable. If we try to make the kink wider, then more of the field will be sitting on the hump, raising the energy. If we try to squeeze it into a smaller region, then the curvature goes up, raising the energy. There is an energetically favorable shape which we found by integrating the Euler-Lagrange equations. This situation is similar to a chain (with some tension) lying on a humped surface in a gravitational field.[20] Note that we can put the kink anywhere by changing b, but that with the imposed boundary conditions it must be somewhere. We can see that this is not changed by letting things vary with time. In fact, the Lorentz invariance of the Lagrangian implies that if $\phi(x,t)=f(x)$ is a time independent solution, then $\phi(x,t)=f\left(\frac{x-vt}{\sqrt{1-v^2}}\right)$ is also a solution for $v<1$. The kinks can move like particles.

The important thing to notice about this example is that from the given structure of the problem and the assumption that there is no energy at infinity, our solutions split into topological classes that are not connected by a continuous

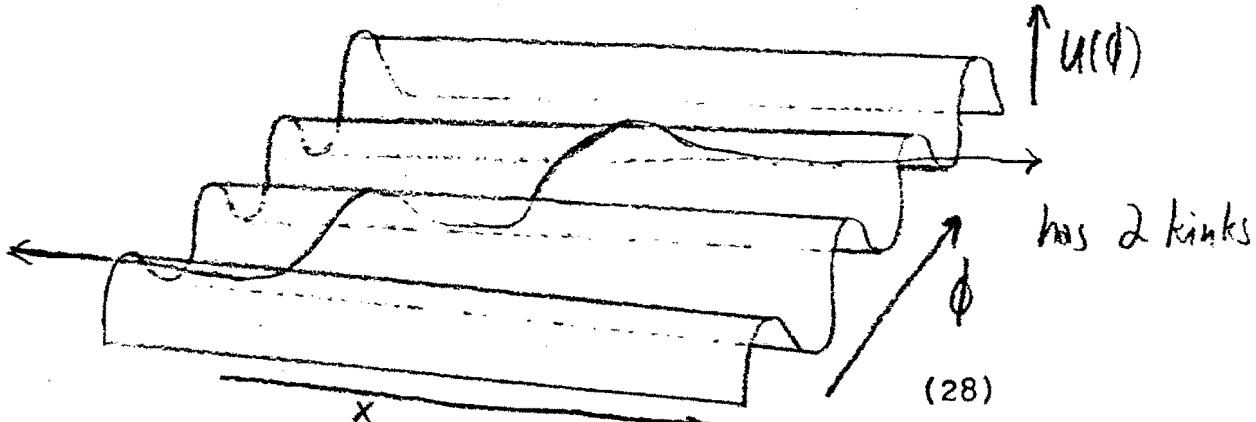
transformation (thus no physical process can mix them).

In the given example we have 4 such classes:



The first two must have a kink and anti-kink respectively, regardless of the form of the field inside the rectangle. In this example we can have widely separated kink, anti-kink pairs but the net kink number=#kinks - #anti-kinks is a conserved quantity. This is an example of a topological conservation law which, in the quantum case, would allow us to define "topological quantum numbers".

The ϕ^4 potential example could only have one net kink or anti-kink because $U(\phi)$ had only two minima. We can easily get any number of net kinks by considering $U(\phi)$ with many minima. One popular example is known as the sine-gordan equation: $U=\frac{\alpha}{\beta^2}(1-\cos\beta\phi)$ leading to the equation of motion: $\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+\frac{\alpha}{\beta}\sin\beta\phi=0$ (the Klein-Gordan equation: $\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+m^2\phi=0$ was one of the first attempts at a relativistically invariant quantum theory). This clearly can have solutions with any number of kinks:



This example is particularly interesting since in 1975 Coleman showed that the quantum version of the sine-gordan equation is equivalent to the zero-charge sector of the massive Thirring model. The soliton conservation of the sine-gordan equation goes over to conventional kinematical Noetherian conservation of the number of quanta of the Thirring model field.[22]

These ideas arise quite naturally in the somewhat artificial case of one-dimensional fields. Unfortunately, the extension to higher dimensions is not straightforward due to a result known as Derrick's Theorem. For a theory, such as we have been considering, with Lagrangian density: $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi)$ where $U(\phi)$ has a minimum value of zero, in D spatial dimensions, the only time-independent solutions are the ground states, for $D \geq 2$.

It is intuitively clear that we cannot get things trapped like we did before in one dimension, because outside of some large sphere the energy must vanish. Since, for $D \geq 2$, this region is connected, the value of ϕ must lie entirely in one component of the zero-set of U. Thus we can continuously distort ϕ within the sphere to lie entirely in one of the zeroes of U.

The proof that there are no time independent solutions is an interesting scaling argument, though. We define: $V_1[\phi] = \frac{1}{2}\int (\vec{\nabla}\phi)^2 d^D\vec{x}$ and: $V_2[\phi] = \int U(\phi)\, d^D\vec{x}$. So the action per unit time is :

$$L[\phi] = \int \mathcal{L}(\vec{x})\, d^D\vec{x} = \int \left\{ \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - U(\phi) \right\} d^D\vec{x} = -V_1[\phi] - V_2[\phi]$$

since we're considering time independent solutions. We want the action to be stationary: L=0. We consider variations of ϕ parameterized by λ : $\phi_\lambda(\vec{x}) \equiv \phi_1(\lambda\vec{x})$ where $\phi_1(\vec{x})$ is the extremal solution. Now $d^D x = \lambda^{-D} d^D(\lambda\vec{x})$ and $\frac{\partial}{\partial x} = \frac{\partial(\lambda x)}{\partial x}\frac{\partial}{\partial(\lambda x)} = \lambda \frac{\partial}{\partial(\lambda x)}$ so we see:

$$V_1[\phi_\lambda] = \tfrac{1}{2}\int (\vec{\nabla}\phi_\lambda)^2 d^D\vec{x} = \tfrac{1}{2}\int (\lambda \vec{\nabla}_{(\lambda\vec{x})}\phi_\lambda)^2 \lambda^{-D} d^D(\lambda\vec{x}) = \lambda^{2-D} V_1[\phi_1]$$

$$V_2[\phi_\lambda] = \int U(\phi_\lambda) d^D\vec{x} = \lambda^{-D}\int U(\phi_\lambda) d^D(\lambda\vec{x}) = \lambda^{-D} V_2[\phi_1]$$

by change of variables in the integrals. So:

$$L[\phi_\lambda] = -\lambda^{2-D} V_1[\phi_1] - \lambda^{-D} V_2[\phi_1]$$

$$\delta L = \frac{d}{d\lambda} L[\phi_\lambda]\Big|_{\lambda=1} = 0 = \left[-(2-D)\lambda^{1-D} V_1[\phi_1] + D\lambda^{-D-1} V_2[\phi_1]\right]_{\lambda=1}$$

$$= (D-2)V_1[\phi_1] + D V_2[\phi_1] = 0$$

but V_1 and V_2 are positive if ϕ_1 is not identically a zero of U. Thus for $D \geq 2$ we cannot have any time-independent solutions of finite energy.[23]

Thus for higher dimensions we need a more complicated field structure. In the next section we discuss just such a structure which has appeared in a number of different places in physics.

V. Gauge Theories

At the present time, the fundamental theories that hold ut the most promise for a unified understanding of physical nteractions are probably gauge theories. The basic idea ehind these theories is the introduction of new fields to nforce the local invariance under some previously known lobal symmetry. Some of the basic concepts involved were eveloped in the differential geometry of Riemman around 861.[24] The idea of local gauge transformations was invented y Hermann Weyl in 1918.[25] He applied the idea to electro- agnetism in 1929[25] and in 1951 Schwinger used it in quantum lectrodynamics.[26] In 1954, Yang and Mills developed a non- belian gauge theory for the SU(2) of isospin.[27] In 1967 Weinberg nd Salam[28] invented their unified gauge theory of weak and lectromagnetic interactions, which was shown in 1971 to be renormalizable by 't Hooft.[29] In 1972 gauge theories were pplied to the strong interactions in the theory known as uantum chromodynamics.[30] Einstein's general theory of rela- tivity is basically a gauge theory (see, though, the controversy n this point: Rd. 31). At the present time there are a num- ber of attempts under way to develop Grand Unified TheorieS.

Starting with a known Lagrangian with some global symmetry (apply the same transformation everywhere and the theory doesn't change) we introduce gauge fields and their inter- actions in the following manner:

1. Start with a theory with global symmetry

2. Hypothesize that the symmetry should be local

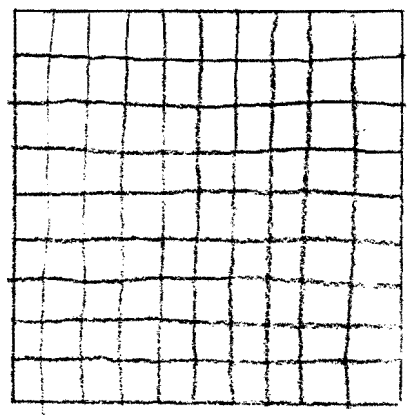
3. Introduce gauge fields to fix up non-symmetrical derivative terms (i.e. replace by covariant derivative)

4. Observe that there is more freedom in the gauge field now than is needed to fix the symmetry

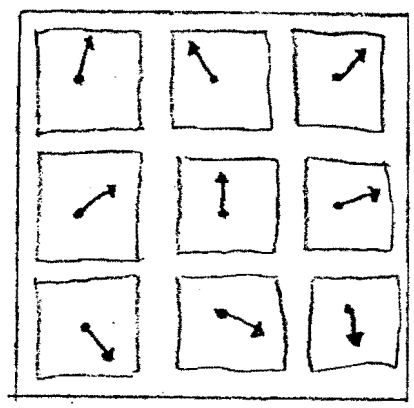
5. Identify the "curvature" field (deviation from "flat", zero-gauge theory)

6. Add a curvature term to the Lagrangian representing the self interaction of the gauge field.

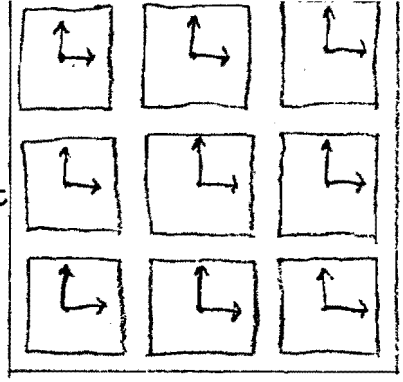
We can get some intuitive feel for this proceedure by looking at a 2-dimensional vector field on a two-dimensional surface. Begin with the theory formulated in terms of an orthogonal coordinate system on a 2-dimensional plane: ⟶



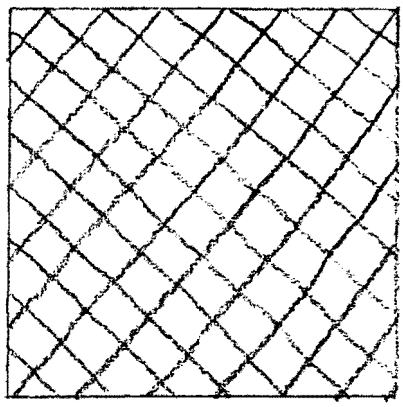
The vectors of the field sit in 2-dimensional tangent vector spaces erected over each point of space: ⟶
(the tangent planes are linear approximations to the base space, in this case they are exact copies.)



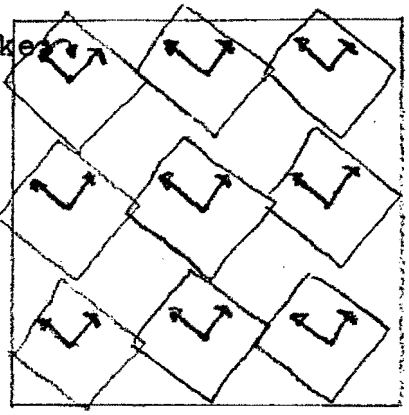
We can choose a basis in each little tangent space to correspond to the coordinate system on the plane: ⟶



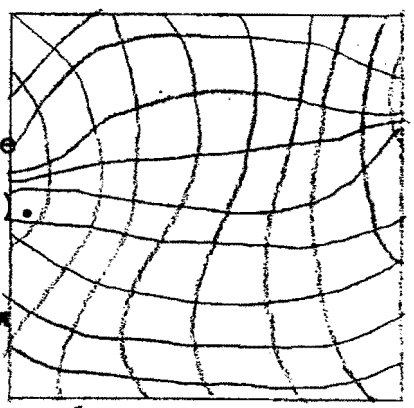
Our theory shouldn't pick out a particular direction in space, so it should have the same form if we rotate our coordinate system: ⟶



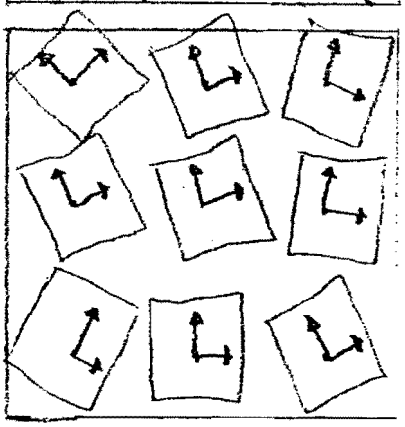
The new bases in the tangent spaces look like: We have changed things at each place in space in exactly the same way, so this is a global symmetry. The symmetry group is the two dimensional rotation group SO(2).



We now claim that our choice of coordinate system here at Stanford, shouldn't affect what they use at Berkeley. We should be able to make arbitrary choices of coordinate orientation (as long as they are continuous). We should be able to express our theory in terms of arbitrary curvilinear coordinates.



And our tangent planes' bases can rotate wildly: ⟶



Anything which depends on the field only within a given tangent plane (i.e. at a given point) will still work fine since each individual plane has only undergone the same transformation it would have in some global rotation. But now consider what happens if we have to compare quantities at different points. Under all these crazy coordinate system changes, the vectors of the field haven't budged a bit, but their descriptions have changed all over the place. When we were doing global transformations, all the descriptions were changing in the same way, so we could still compare descriptions at different points. In the present case, though, if we try to compare our vector field at two different points we'll get a piece due to the actual change in the vector field, but also another piece due to the change of the description system. Telling how to compare a given point with an arbitrary point would be a horrendous job. Telling how to compare a given point with one infinitesimally close, however, is easy since all changes are continuous and we can work to first order in the distance. What we need is two matrices at each point. One tells us how much our coordinates rotate when we go in the x direction, one in the y direction. Since we're moving infinitesimally, we want these matrices to be the derivative of the coordinate rotation with respect to distance. Our matrices are thus elements of the Lie algebra of the transformation group. Specification in these two directions is enough because everything is linear.

We can combine these matrices into an object that looks

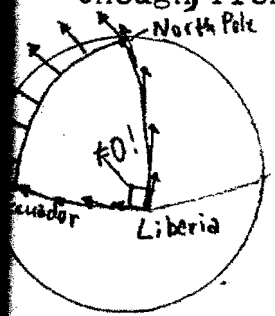
something like a third rank tensor field, with one index for the direction in space and the other two giving the component of the matrix. The transformation properties cannot be those of a tensor, though, because we get an inhomogeneous additive piece every time the transformation changes from point to point (consider how they change going from orthogonal coordinates to curvilinear ones). In differential geometry these objects are called connections and are often written as the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$. $\Gamma^{\alpha}_{\beta\gamma}$ is the $\alpha\beta$ component of the change in the βth coordinate basis when you move in the γth direction.[32] In the general case, the objects corresponding to the connections will be the gauge field. With the connections we can define a new kind of derivative, called the covariant derivative D_μ, that takes into account the fact that the coordinates are changing and only gives the actual change of the vector field itself.

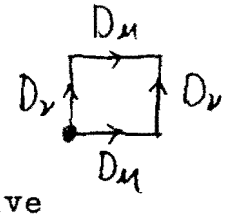
The reason we had to do all this was that the theory as expressed before with ordinary derivatives (change in coordinate values) was not invariant under the local transformations. When we replace all the old derivatives by covariant derivatives the theory becomes invariant under these transformations.

Now we notice to our chagrin that the connections have more freedom in them than we desired. But, surprisingly, this extra freedom is in some sense necessary. Nothing we can say about the connections at a given point can prevent it. It depends on the way the connections change from point to point.

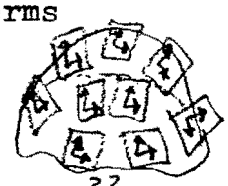
Consider a vector at a given point. Define a new vector at each point along a path away from the initial point in such a way that the covariant derivative of these vectors along the path always vanishes. This is called parallel transport. Now take the path around in a loop to end up at the point where we started. In the nice flat space that we started with only one thing can happen: the vector at the end of the path must point in the same direction as the one we started with (after all, the vectors are parallel).

But, surprisingly enough, we can find connections where the final vector points in another direction from the first. These must describe a situation with more in it than just our plane and vector field. This sort of thing is familiar, though, from cartography. If we start with a vector pointing northward in Liberia and slide it along a meridian, always keeping it "parallel" to itself (but tangent to the earth), up to the north pole and then down another meridian to Ecuador, and then along the equator back to Liberia, we find our vector rotated by 90 degrees. This effect is due to the curvature of the surface. In fact, we can define the curvature at a point by a limiting case of this process. Consider a little square at the point. In the limit that the area of the square vanishes, define the amount that vectors get rotated by traversing the square divided by the area to be the components of a curvature tensor in the "direction" of the square (i.e. an index for the components of each of the legs of the square). In the limit





this is just the commutator of the covariant derivative: $F_{\mu\nu} = [D_\mu, D_\nu]$. In differential geometry this is a derivative of a rotation and so a 2×2 matrix (for each μ, ν). The entire four-index tensor $R^{\alpha}_{\beta\mu\nu}$ is the Riemmanian curvature tensor. We have a new field which we must write the interaction terms for. We'll see a gauge invariant way that extremizes the total curvature (a little like soap bubbles).



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Let's see how this works for the case of electromagnetism. For definiteness, we begin with a fermion field $\psi(x)$ with Lagrangian: $\mathcal{L} = -i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$ (we might just as well have used the Schrödinger wavefunction). We notice that the Lagrangian is invariant under a global phase transformation: $\psi \to e^{i\theta}\psi$ where $\bar{\psi} \to \bar{\psi}e^{-i\theta}$ (θ constant over all spacetime). Thus we have a global U(1) symmetry. We hypothesize that we should be able to choose the phase (coordinate axes in the complex plane) arbitrarily at widely separated points. We suggest that the Lagrangian <u>should</u> be symmetric under the local phase transformation: $\psi \to e^{i\theta(x)}\psi$, as long as θ is a smooth function of x. The $-m\bar{\psi}\psi$ term is fine since $-m(\bar{\psi}e^{-i\theta(x)})(e^{i\theta(x)}\psi) = -m\bar{\psi}\psi$ but the derivative term causes problems since θ is changing with x: $-i(\bar{\psi}e^{-i\theta(x)})\gamma^\mu\partial_\mu(e^{i\theta(x)}\psi) = -i\bar{\psi}e^{-i\theta(x)}\gamma^\mu\{i(\partial_\mu\theta(x))e^{i\theta(x)}\psi + e^{i\theta(x)}\partial_\mu\psi\}$ $= -i\bar{\psi}\gamma^\mu\partial_\mu\psi + \bar{\psi}\gamma^\mu\psi(\partial_\mu\theta(x))$. We get an extra piece that messes up the symmetry. As we saw, we may fix this by introducing a connection $A_\mu(x)$ and the corresponding covariant derivative: $D_\mu = \partial_\mu - ieA_\mu(x)$. Our new Lagrangian is $\mathcal{L} = -i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi$ For this to be invariant under the change $\psi \to \psi' = e^{i\theta(x)}\psi$ and $D_\mu \to D'_\mu = \partial_\mu - ieA'_\mu$ we must have: $-i\bar{\psi}\gamma^\mu D_\mu\psi = -i\bar{\psi}e^{-i\theta(x)}\gamma^\mu D'_\mu(e^{i\theta(x)}\psi)$

This implies: $D'_\mu(e^{i\theta(x)}\psi) = e^{i\theta(x)} D_\mu \psi = (\partial_\mu - ie A'_\mu)(e^{i\theta(x)}\psi)$

$= ie(\partial_\mu\theta(x)) e^{i\theta}\psi + e^{i\theta}\partial_\mu\psi - ie A'_\mu e^{i\theta}\psi = e^{i\theta}(\partial_\mu - ie(A'_\mu - \partial_\mu\theta))\psi$

$= e^{i\theta} D_\mu \psi = e^{i\theta}(\partial_\mu - ie A_\mu)\psi \Rightarrow A_\mu = A'_\mu - \partial_\mu\theta \Rightarrow A'_\mu = A_\mu + \partial_\mu\theta$

We've seen a field theory with this kind of symmetry. The electromagnetic potentials A_μ (fortuitous notation) give the same physics under what is known in E&M as a gauge transformation (fortuitous name), which has exactly this form. (In differential-form notation: the physics depends on F=dA so making A into A+dθ doesn't change anything since ddθ=0 always).

Now consider the "curvature" term which we saw should be the commutator of the covariant derivative. We define:

$F_{\mu\nu} = \frac{i}{e}[D_\mu, D_\nu] = \frac{i}{e}[(\partial_\mu - ieA_\mu), (\partial_\nu - ieA_\nu)] = \frac{i}{e}(ie\partial_\nu A_\mu - ie\partial_\mu A_\nu) = \partial_\mu A_\nu - \partial_\nu A_\mu$

Amazingly the commutator of these two operators is no longer an operator. Even more amazingly, the $F_{\mu\nu}$ thus defined is exactly the tensor "Faraday" which contains the electric and magnetic fields. Notice that in this abelian (elements of the group U(1) commute) case that $F_{\mu\nu}$ is itself invariant:

$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu + \partial_\nu\theta) - \partial_\nu(A_\mu + \partial_\mu\theta) = \partial_\mu A_\nu + \partial_\mu\partial_\nu\theta - \partial_\nu A_\mu - \partial_\nu\partial_\mu\theta = \partial_\mu A_\nu - \partial_\nu A_\mu =$

To account for the energy in the field we add the Lorentz invariant term: $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. The total Lagrangian becomes:

$\mathcal{L} = -i\bar\psi\gamma^\mu D_\mu\psi - m\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi + e\bar\psi\gamma^\mu\psi A_\mu - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu$

The Euler-Lagrange equations are thus:

$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = 0 = \partial_\mu(-i\bar\psi\gamma^\mu) + m\bar\psi - e\bar\psi\gamma^\mu A_\mu = -i(\partial_\mu - ieA_\mu)\bar\psi\gamma^\mu + m\bar\psi = 0$

$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)} - \frac{\partial\mathcal{L}}{\partial\bar\psi} = 0 = i\gamma^\mu\partial_\mu\psi + m\psi - e\gamma^\mu\psi A_\mu = -i\gamma^\mu(\partial_\mu - ieA_\mu)\psi + m\psi = 0$

$\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)} - \frac{\partial\mathcal{L}}{\partial A_\beta} = 0 = \partial_\alpha\left(-\frac{1}{4}(2\partial_\alpha A_\beta - \partial^\beta A^\alpha - \partial_\beta A_\alpha - \partial^\beta A^\alpha - \partial_\beta A_\alpha + 2\partial_\alpha A_\beta\right) - e\bar\psi\gamma^\beta\psi$

$= \partial_\alpha(\partial^\beta A^\alpha - \partial^\alpha A^\beta) - e\bar\psi\gamma^\beta\psi \Rightarrow \partial_\alpha F^{\alpha\beta} = e\bar\psi\gamma^\beta\psi = ej^\beta$ with $j^\beta = \bar\psi$

The first two of these are the equations of motion for a charged particle in an electromagnetic field. The last is the 2 Maxwell source equations (the other two are the Bianchi identities for curvature) with the fermion field identified as the charge-current source. When quantized the gauge field gives us photons, the intermediator of the electromagnetic force.

It was the idea of Yang and Mills to extend this procedure to a bigger group than U(1).[34] Consider a gauge group G and a multiplet of fields $\psi(x)=\begin{pmatrix}\psi_1(x)\\ \vdots \\ \psi_n(x)\end{pmatrix}$ which transform into one another under an appropriate matrix representation of G. We want our Lagrangian to be symmetric under the transformation $\psi(x)\to\omega(x)\psi(x)$ where $\omega(x)$ is an element of this representation of G. We introduce the gauge field $W_\mu(x)$ (μ is a Lorentz index and W is in a representation of the Lie algebra of G) and the accompanying covariant derivative $D_\mu=\partial_\mu-ieW_\mu$. Again we want D_μ to transform like $D'_\mu(\omega\psi)=\omega D_\mu\psi$. Because our group is possibly non-abelian, the gauge field transforms like: $W'_\mu=\omega W_\mu\omega^{-1}-\frac{i}{e}(\partial_\mu\omega)\omega^{-1}$. The Yang-Mills field strengths (curvature terms) are: $G_{\mu\nu}=\frac{i}{e}[D_\mu,D_\nu]=\frac{i}{e}[(\partial_\mu-ieW_\mu),(\partial_\nu-ieW_\nu)]=\partial_\mu W_\nu-\partial_\nu W_\mu+\frac{e}{i}[W_\mu,W_\nu]$ and after a tedious calculation we see it transforms like: $G'_{\mu\nu}=\omega G_{\mu\nu}\omega^{-1}$. These results all reduce to what we had before for the abelian U(1) case.

We can pick a particular basis of group generators σ_a for the Lie algegra (a runs from 1 to the dimension of the Lie algebra) and express $W_\mu=\frac{1}{2}\sigma_a W^a_\mu$ and $G_{\mu\nu}=\frac{1}{2}\sigma_a G^a_{\mu\nu}$ where W^a_μ and $G^a_{\mu\nu}$ are just numbers now. If the structure constants

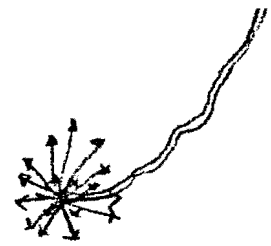
of the Lie algebra are f_{abc} (from $[\sigma_a, \sigma_b] = f_{abc}\sigma_c$) then $G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e f_{abc} W^b_\mu W^c_\nu$ From the Lagrangian: $\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$ we obtain equations of motion: $\partial^\nu G^a_{\mu\nu} = e f_{abc} G^b_{\mu\nu} W^\nu_c$. Note that if our group is non-abelian (so the f_{abc} aren't identically zero), these field equations are nonlinear (because the intermediate bosons can carry a gauge charge themselves) and explicitly dependent on the gauge field W. When the theory is quantized we get a new kind of gauge particle for each generator in the group. Thus the electromagnetic U(1) gives the single photon, a weak SU(2) gives 3 (the 3 Pauli matrices are a typical basis for SU(2)'s Lie algegra) intermediate vector bosons (actually these are tangled up with the photon but the total number is the same), and the QCD strong SU(3) gives 8 gluons (elements of the Lie algebra of a group of unit-determinant unitary matrices are themselves hermitian and traceless, so from the 18 possible degrees of freedom in the 9 complex entries we lose 3 real and 6 imaginary for hermiticity and 1 more to be traceless). In a pure gauge theory we cannot add a mass term for the gauge field to the Lagrangian since it is not gauge invariant (remember the inhomogeneous piece in its transformation), and so these particles have to be massless.

V. Monopoles and Instantons

We now have enough of an apparatus to build some topological excitations in more than one dimension. The objects that we'll discuss in this section are all found in classical field theories. To date, no objects of this sort have been observed experimentally (not so for their statistical mechanical counterparts discussed in the next section). This is discouraging but the ideas involved are intriguing and some variant of them might prove important in the future.

There appear to be two ways for topology to non-trivially enter into our physics. The first occurs when the base manifold that we're working on (eg. spacetime or phase space) has a non-trivial topological structure. In the case of spacetime, experimentally this can only apply to the very large or very small structure of the universe. The other way is to consider the topological structure of the space of solutions in our theory. This usually arises from the boundary conditions on our solutions at infinity.

As an example of the first type let us consider the magnetic monopole. At first we might think that the assumption of an electromagnetic potential (which appears to be justified by the Bohm-Aharonov effect which shows that the potential has a real quantum effect modulo a gauge transformation[35]) precludes a source of magnetic flux. Remember that the $\vec{\nabla}\cdot\vec{B}=0$ Maxwell equation is an identity if $\vec{B}=\vec{\nabla}\times A$. Dirac got out of this difficulty by letting the potential have a point singularity at every radius. These may be



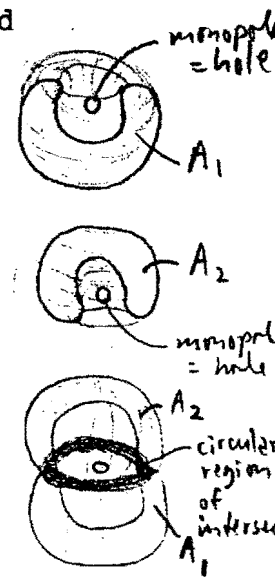
connected together into a "string" singularity eminating from the monopole. One may think of the string as a thin, long solenoid going off to infinity and any flux that it contains naturally comes out the end looking like a monopole. The requirement that the string not be observable led Dirac to his famous monopole quantization condition $\frac{2ge}{\hbar c}$ =integer,[36] if g is the magnetic charge on the monopole.

In 1975 Yang and Wu[37] applied some concepts from fiber-bundle theory to this problem and found a new way to get Dirac's result without his string. The basic idea behind a fiber-bundle is that we have quantities on a manifold which are not (and in non-trivial cases cannot be) defined globally. Instead they are defined on little patches that cover the whole manifold. Where patches overlap they may give different values to the quantities, but they are related by some definite transformation. This type of structure is quite natural for gauge theories where things are defined only up to a gauge. In general, Bohm-Aharanov type experiments hint that some quantities should have path dependent values.

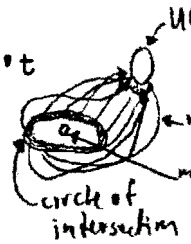
We may understand the Wu/Yang monopole as follows: We want the integral of the radial component of $\vec{B}$ over a sphere S that surrounds the monopole to be non-zero. Since the integral is taken at a given time, any vector tangent to S will give zero when plugged into (contracted with) the one-form dt. Thus $dt \wedge dx$, for instance, gives zero when it acts on a two-vector element of surface area of S. So the integral over S of the normal component of $\vec{B}$ is the same as

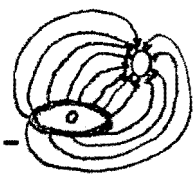
the integral over S of the two-form $F=E_x dx\wedge dt+E_y dy\wedge dt+E_z dz\wedge dt+$ $+B_x dy\wedge dz+B_y dz\wedge dx+B_z dx\wedge dy$. But Stokes theorem says that $\int_{S=\partial T} F=\int_T dF=\int_T ddA=0$ if F can be written as dA within the sphere. Now consider removing the point of the manifold where the monopole is. This makes the second homology group non-trivial (piece together four triangles around the hole so that they have no boundary, yet are not the boundary of a tetrahedron because of the hole). De Rham cohomology tells us that this implies the second cohomology group is also non-trivial. This is the group of closed 2-forms (d acting on them gives 0) modulo the group of exact 2-forms (they are themselves d acting on some other form). In fact, the elements of the group can be labelled by the integral over the chain of triangles that gave us a non-trivial homology group. This is just the integral over the sphere that we started with, i.e. the net outward magnetic flux.

All that we have said is that we can build a monopole by removing a point from the manifold but that we have to give up hope of finding a single potential that is defined everywhere. Consider, though, a hemispherical region that surrounds the monopole from below. Because there are no holes in this region and dF=0 everywhere within it, we can find an A_1 so that $F=dA_1$. Similarly we can find an A_2 so that $F=dA_2$ in a hemisphere above the monopole. Now consider the two halves together. In the circular region of overlap the two potentials A_1 and A_2 give the same field F. They must therefore be related

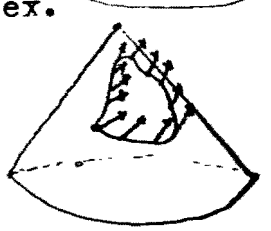


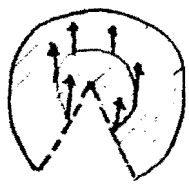
by a gauge transformation at each point around the circle. Remember that for electromagnetism the gauge group is U(1) which looks like a circle topologically. Thus we want a map from the circle of intersect of the two regions into the circle of the gauge group. If we choose a map that doesn't wind around the gauge group then it is homotopic to the map where everything goes to the identity of the group and so by a continuous gauge transformation we can make A_1 and A_2 agree everywhere. We have seen that this implies no source of B. The monopole comes from the cases where we wrap once around, twice around, etc. Clearly no continuous gauge transformation can change the homotopy class of the map. Thus the topological character of Dirac's quantization is evident.





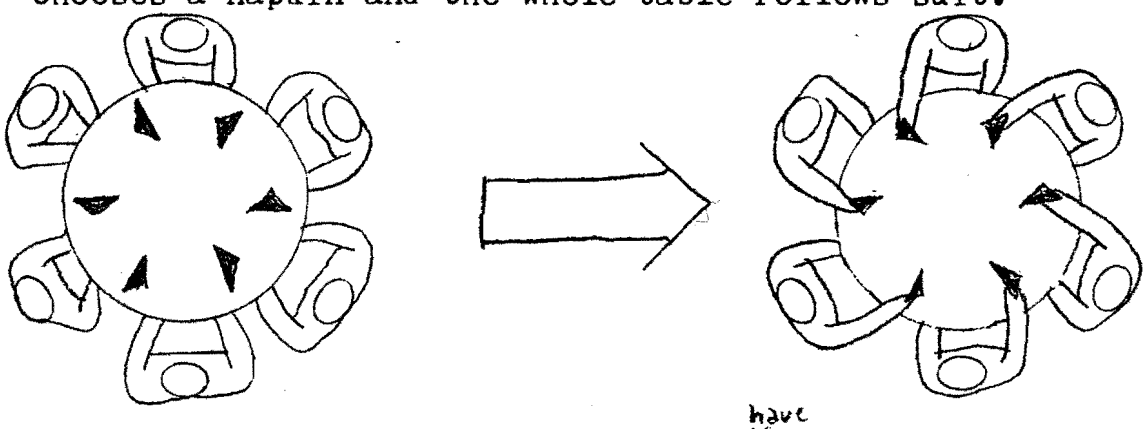
What does it mean to remove a point from the manifold? One might consider this a singularity in the field or a region where our description breaks down, but it is interesting to contemplate actually removing a point from space. This is the sort of picture one is led to by the concept of black holes. There is another concept that this example suggests. We got rid of the "curvature" of a field that has a net "curvature" by putting it all at the hole. We may do the same thing for curvature of space. In two dimensions we can mock up any curvature arbitrarily closely by a space which is everywhere flat but has a gas of holes. By removing a single point, for instance, our space can be everywhere flat but in the shape of a cone with the hole at the vertex. If we don't go around the vertex then vectors will point





in the same direction after going around a loop. If we go around the vertex, however, they get rotated. We can see this by undoing the cone and spreading it out on a plane where we know what parallel means.

Let's now consider the excitations that get trapped by boundary conditions. These will be generalizations to higher dimensions of the one dimensional fields considered earlier. They have in common the feature of spontaneous symmetry breaking. This label is applied when the Lagrangian of some theory possesses a symmetry that is not possessed by the ground state (vacuum). Abdus Salam gives the example:[39] At an international dinner, where there are no set conventions of etiquette, a group is seated about a circular table with a napkin between each pair of people. The situation is symmetric with respect to left and right hands until one person chooses a napkin and the whole table follows suit.



In quantum field theory we would have operators that are the infinitesimal generators for some symmetry of the Lagrangian that don't annnihilate the vacuum state $|0\rangle$. One way to do this is to introduce the so-called Higgs field Φ which has a non-zero vacuum expectation value $\langle 0|\Phi|0\rangle \neq 0$. Classically,

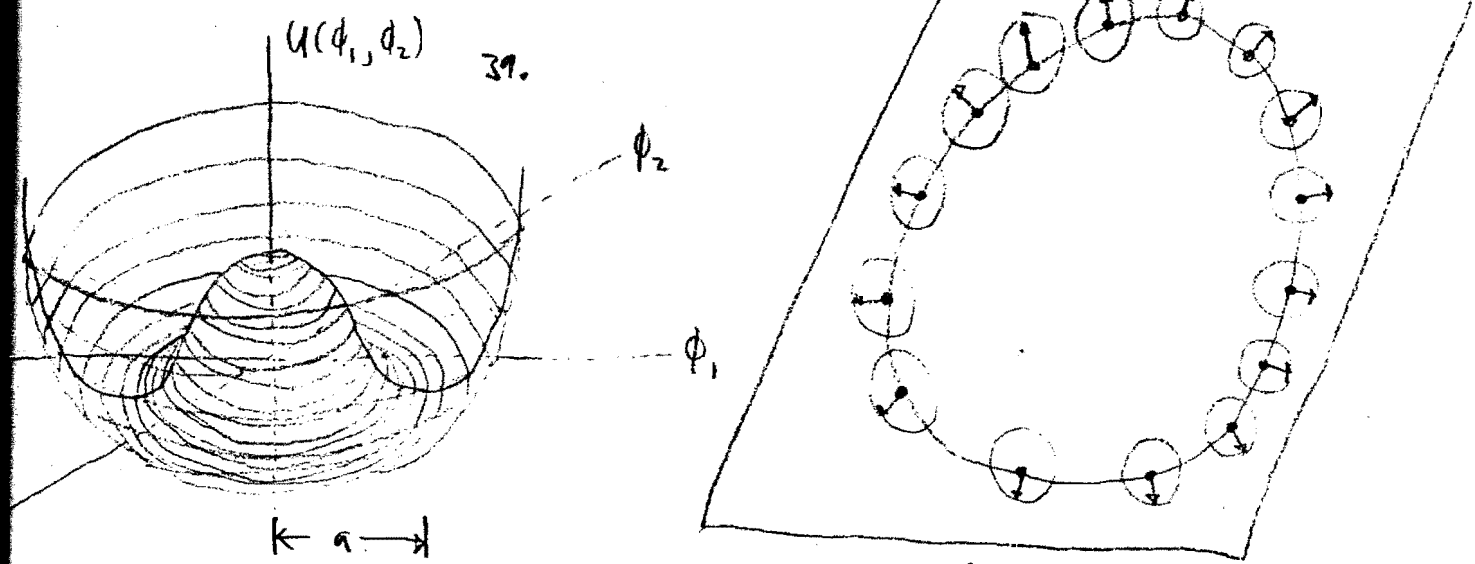
it is the Higgs field whose potential energy term has more than one minima (in general we will have a continuum of ground states, as opposed to the discrete set we saw in the sine-gordan equation).

We will consider models which have a gauge symmetry that is broken. This is an important case since it is through this mechanism that gauge particles can acquire a mass. We will call our gauge group G. Assume that all minimum energy field configurations are related to one another by an element of the gauge group (so there are no "accidental" minima not arising from the gauge symmetry). In general, only a subgroup of the full gauge group G will take us from one ground state to another one (eg. in the unbroken case where there is one ground state, the entire gauge group takes the ground state to itself). Call the subgroup of G that doesn't change the ground state, H. Thus even after picking a particular ground state, H is still a good symmetry. From a given ground state elements of the quotient group G/H will thus take us to every other ground state. We can associate elements of G/H with ground states. Since "close" group elements take us to "close" ground states, the topological structure of the space of ground states is isomorphic to G/H.

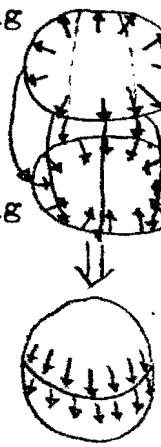
The basic idea behind topological excitations in this setting is that the assumption of no energy at infinity means that on a large enough sphere (i.e. at infinity) the field configuration must be one of the ground states. If there are topologically non-trivial maps from this sphere into G/H

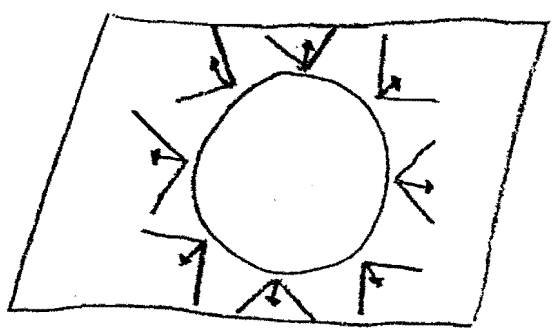
then there will be topologically distinct classes of solutions. Sometimes this forces some energy to be localized somewhere inside the sphere (in the same way that the sine-gordan boundary conditions could trap energy when the field had to go over a hump in the potential). The reason that we had to consider gauge theories to get this sort of thing is that we have to have a symmetry that we can utilize at each point on the sphere independently.

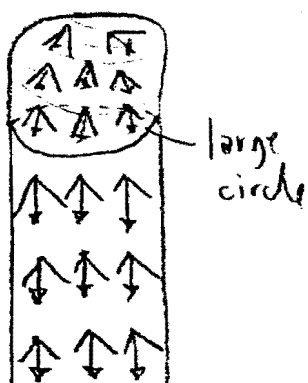
There are an infinite number of examples which we might consider but I'll restrict them to a few representative ones. 40. Let us consider a two dimensional space with a two dimensional field $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$, like our first model for a gauge theory. Our gauge group will again be SO(2). We want a potential which has minima related to one another by the gauge group. We consider the "sombrero" shaped potential $U=(\phi_1^2+\phi_2^2-a^2)^2$. Here H is just the identity element and the space of ground states is isomorphic topologically to G/H=G=SO(2), i.e. the circle S^1. (which sits at the bottom of the sombrero). Consider a big circle in the plane with a large enough radius that there is no energy there. We want to choose



elements of G/H there. We know that $S^1 \to S^1$ can be topologically nontrivial since $\pi_1(S^1)=Z$. Let us consider the one where as we go around the big circle, we go around the base of the sombrero once. So we have little vectors (of length a) pointing radially outward. But now notice that with this boundary condition the vector field must vanish somewhere in the circle. If it didn't we could make another one with the vectors pointing inward, cut out the big circles, and paste them edge to edge and we would have a continuous non-zero vector field on a two dimensional sphere, contrary to the Brouwer theorem we mentioned earlier. But if the field vectors have a length of less than a we get some energy (we ride up on the hump of the sombrero). Thus we necessarily localize some energy inside the sphere. "But wait", we say, "We're supposed to have zero energy at infinity. We get energy from changes in the field and it looks like it's rotating wildly out there." Remember, though, that this is a gauge theory and that the energy term uses the covariant derivative. When we drew the picture we used rotated frames (they turned when we made the gauge transformation). In the correct frames the vectors aren't changing at all. Now we notice another interesting consequence: if we warp our surface in three dimensions so that covariant derivatives look like ordinary derivatives, the piece from the large circle outward must roll into a long tube, since the







coordinates rotate by 2π by going around the circle. Somewhere inside the circle we must cap off the cylinder. We can see that this requires curvature in our gauge field. Thus we have forced some energy into the gauge field as well by these boundary conditions. This model is not so far-fetched as it might seem. If we stack a bunch of these planes on top of one another the energy packets we found together form a flux line in the Landau-Ginzburg theory of Type II superconductors (the "curvature" in the gauge field is the magnetic flux passing through the plane).

Our other examples are trivial extensions of the above. We might consider a field with three components $\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$ and with gauge group SO(3). If we use the 3-dimensional version of the sombrero: $U=(\phi_1^2+\phi_2^2+\phi_3^2-a^2)^2$ then the ground states may be identified with points on a sphere ($\phi_1^2+\phi_2^2+\phi_3^2=a^2$). Now the unbroken symmetry is not trivial, though, because once we have picked out a point on this sphere we can still rotate about the axis it determines (so H=SO(2)). We have just said that topologically: $G/H=SO(3)/SO(2)\sim S^2$. If we are on a two dimensional plane then the map from the circle at infinity to G/H is trivial (we saw $\pi_1(S^2)$= the trivial group). Consider what happens, though, if the field is defined on three dimensional space. The sphere at infinity is S^2 and we saw that $\pi_1(S^2)=Z$ is not trivial. The non-trivial solutions here are similar to the original Polyakov "hedgehog" or 't Hooft "monopole" found in 1975[41] and 1974[42] (they used SU(2) which we saw was the covering space of SO(3) and $G/H=SU(2)/U(1)\sim S^2$).

The one last example[43] we'll consider is a field in Euclidean spacetime (i.e. metric $\propto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which we may interpret as ordinary Minkowski space with imaginary time). The sphere at infinity is S^3. If we have the gauge group SU(2), which recall we showed was also topologically S^3, then since $\pi_3(S^3)=Z$ we get non-trivial maps. The "objects" that result are called instantons. They are events that represent transitions from one ground state to another (quantum mechanically, tunnelling from one vacuum to another). Thus the Yang-Mills vacuum could be a complex beast seething with instantons. If we had kept an unbroken U(1) so $G/H=SU(2)/U(1)=S^2$ (imbed U(1) like $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$), then we could have used the non-trivial Hopf map $\pi_3(S^2)=Z$ that we showed earlier.

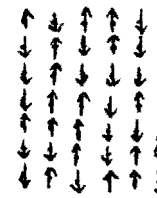
VI. The Ising model and Kosterlitz-Thouless vortices

Let's now shift gears and consider some statistical mechanics. Probably the most fundamental problem in this field is understanding in detail the emergence of high-level order from low-level phenomena, i.e. the problem of phase transitions and critical phenomena. This is of course crucial to the understanding of biological phenomena and the great diversity of the different regimes found in the universe in general. We will examine how some ideas similar to the topological ones we have been considering affect that understanding.

Probably our greatest understanding of phase transitions to date has come from studies of the Ising model and related systems. This model was invented in 1925[44.] by Ising as a model for ferromagnetism. This is the observed phenomena of spontaneous magnetization below some critical temperature in certain materials. It arises from alignment of the electronic spin of atoms in a regular crystalline lattice with the concomitant constructive addition of their magnetic moments. The energetic desire of electrons in neighboring atoms to align their spins is not due to magnetic dipole-dipole interactions, which in this case are weak, but rather to the Pauli exclusion principle keeping electrons with aligned spins farther apart and thus lowering the Coulomb energy.[45.] This interaction is very short range since it relies on electron overlap in the non-aligned case.

Ising's model is simply a bunch of spins on a regular

lattice (the most common being square, cubical, etc.) which can either point up (in which case define $\mu = 1$) or down ($\mu = -1$). There is an interaction coefficient J so that the total energy is $E = -\frac{1}{2}\sum_i \sum_{nn(i)} J \mu_i \mu_{nn(i)} + \text{const.} = \frac{1}{4} J \sum_i \sum_{nn(i)} (\mu_i - \mu_{nn(i)})^2$ where μ_i is the spin ($=\pm 1$) at position i and $\sum_{nn(i)}$ is a sum over the nearest neighbors of i in the lattice. This same model serves for modelling binary alloys and lattice gases (where up corresponds to one type of atom and down the other or up to the presence of a gas molecule and down to its abscence). To show some of the features of this type of model we will investigate it perturbatively in both the low temperature and high temperature limits. The appropriate quantity to investigate is the partition function: $Z = \sum_{\text{states}} e^{-\beta \mathcal{E}}$, where $\mathcal{E}$ is the energy of a state.



Let us first consider the one-dimensional Ising model, ↑↓↓↑↑↑↓↑↓ with the boundary condition that the spins on the end point up so that when the temperature is zero all spins will point upward. The partition function in this case is given by:

$$Z = \sum_{\mu_1 = \pm 1, \ldots, \mu_N = +1} \exp\left(-\beta \sum_{i=1}^{N-1} \frac{J}{2} (\mu_i - \mu_{i+1})^2\right)$$

The sum in the exponential is equal to J times the number of flipped bonds. We do a consistent expansion of Z in powers of $\alpha = \exp(-2J)$ which is small when βJ is large or T is small compared to $\frac{J}{k}$. The α^0 term arises when there are no flipped bonds (↑↑↑↑↑↑↑↑). Since we are considering the case where all point up at T=0 this means $Z = 1 + O(\alpha)$. The conditions at the boundary prevent there being an odd number of flipped

bonds. With two flipped bonds (↑↓↑↑↑↑↑ or ↑↓↓↓↓↑↑↑↑ etc.) we get the α^1 terms. The first flip can be at any of the first N-2 bonds, say the jth. The second can be anywhere from j+1 to N-1. The total number is $\sum_{j=1}^{N-2}(N-(j+1))=$ $(N-2)(N-1)-\sum_{j=1}^{N-2} j=(N-2)(N-1)-\frac{1}{2}(N-2)(N-1)=\frac{1}{2}(N-2)(N-1)$. So $Z=1+\frac{1}{2}(N-2)(N-1)+O(\alpha^2)$.

We can investigate the average spin of a given site for the system at equilibrium at a low temperature. We know that the probability of a given configuration is $\frac{1}{Z}\exp(-\beta \times \text{energy of config.})$. Let us find the probability for finding the spin at the center of the chain pointing down. It cannot point down with no flipped bonds. With two flipped bonds it will point down if one flip is on the left and one on the right (↑↑↓↓↓Ⓓ↓↑↑↑↑). The number of ways for this to happen is $\left(\frac{N-1}{2}\right)^2$. So the probability $=\frac{1}{Z}\left(\frac{N-1}{2}\right)^2\alpha+O(\alpha^2)$. If we have four flips then we again need our spin to be in a block of downward spins but we have another block to either the left or right (↑↓↓↑↑Ⓓ↓↓↓↑↑ or ↑↑↑↓↓Ⓤ↑↓↓↓↑). We have already counted the number of ways for a downward block to fit in m spins with the ends pointing up and it $=\frac{1}{2}(m-1)(m-2)$. The total number is thus (with i for the position of the left end of the block with our spin, j for the right)

$$\sum_{i=1}^{\frac{N-1}{2}}\sum_{j=1}^{\frac{N-1}{2}}\left(\tfrac{1}{2}(i-1)(i-2)+\tfrac{1}{2}(j-1)(j-2)\right)=\frac{N-1}{2}\sum_{i=1}^{\frac{N-1}{2}}(i^2-3i+2)$$

We need the expression for $\sum_{i=1}^{n} i^2$. This has a geometric analog of a pyramid where we know the height and want the volume (). We thus expect a third degree polynomial $\sum_{i=1}^{n} i^2=An^3+Bn^2+Cn+D$. Plugging in $n=0\Rightarrow\sum=0$, $n=1\Rightarrow\sum=1$, $n=2\Rightarrow\sum=5$, $n=3\Rightarrow\sum=14$ we find that: $\sum_{i=1}^{n} i^2=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$.

The total number of states is thus:= $\frac{N-1}{2}\left\{\frac{1}{3}\left(\frac{N-1}{2}\right)^3+\frac{1}{2}\left(\frac{N-1}{2}\right)^2+\frac{1}{6}\left(\frac{N-1}{2}\right)-\frac{3}{2}\left(\frac{N-1}{2}\right)\left(\frac{N+1}{2}\right)+2\left(\frac{N}{2}\right.\right.$
$=\left(\frac{N-1}{2}\right)^2\left\{\frac{1}{12}(N^2-2N+1)+\frac{1}{2}(N-1)+\frac{1}{6}-\frac{3}{4}(N+1)+2\right\}=\left(\frac{N-1}{2}\right)^2\left\{\frac{1}{12}N^2-\frac{5}{12}N+1\right\}$

Consistently to order α^2 we have the probability for our spin to be down is$=\left[\left(\frac{N-1}{2}\right)^2\alpha+\left(\frac{N-1}{2}\right)^2\left\{\frac{1}{12}N^2-\frac{5}{12}N+1\right\}\alpha^2+O(\alpha^3)\right]/\left[1+\frac{1}{2}(N-1)(N-2)\alpha+O(\alpha^2)\right]$
$=\frac{1}{4}(N-1)^2\alpha+\frac{1}{4}(N-1)^2\left\{-\frac{5}{12}N^2+\frac{31}{12}N-1\right\}\alpha^2+O(\alpha^3)$

This is clearly wrong for all but $\alpha=0$. If T=0 then $\alpha=0$ and prob.$\downarrow=0$ as we expected. If $\alpha>0$ then our probability depends on N which it should not and as N gets large we get a negative probability. The problem is that the probability is not an analytic function of temperature in this one dimensional case. Exact solutions show that for any $T>0$, the one dimensional Ising model is not magnetized. This occurs because with only the energy to flip two bonds we can flip an arbitrary number of spins. This is not the case in higher dimensions where the energy to flip a block of spins goes up as the surface area of the block.

We now look at the low temperature expansion of Z for the D-dimensional Ising model. For computational simplicity we take our systems to be square lattices on tori (i.e. with periodic boundary conditions). We assume that the system is N bonds wide in each dimension, so the total number of spins is N^D. The partition function is now given by: $Z=\sum_{\mu_i=\pm1,\ldots,\mu_{N^D}=\pm1}\exp\left(\sum_i\sum_{nn(i)}-\beta\frac{J}{4}\left(\mu_i-\mu_{nn(i)}\right)^2\right)$. We wish to expand this consistently in powers of $\alpha=\exp(-2\beta J)$ again (i.e. low temperature). We see that the 1th power of α will have the number of situations with 2l bonds flipped as its coefficient (we can easily see that we can never flip an odd number

of bonds with periodic boundary conditions). Call this number f(l). The partition function is $Z=\sum_{l=0}^{\infty} f(l)\alpha^l$. Let's work out Z to order α^{2D}. Again assuming that at T=0 we have all spins pointing up we see that f(0)=1 since there is one way to have no bonds broken. If we flip one spin we break 2D bonds so the next non-zero f is $f(D)=N^D$ since we could flip any one of the N^D spins. The next term comes from flipping two spins right next to one another. This breaks 2(2D-1) bonds. The first flip could have been any of the N^D and the second any of 2D. So $f(2D-1)=N^D D$ (we divided by two to keep from counting bonds twice). The next term comes from flipping two non-adjoining spins with 2(2D) broken bonds. The first one could be any of the N^D, the second any of the N^D but the 2D+1 adjoining or being the first flip. So $f(2D)=\frac{1}{2}N^D(N^D-2D-1)$ (again the $\frac{1}{2}$ to avoid double counting). So the final expression is $Z=1+N^D\alpha^D+N^D D\alpha^{2D-1}+\frac{1}{2}N^D(N^D-2D-1)\alpha^{2D}+O(\alpha^{2D+1})$

We can find the probability of finding a given spin pointing down to order α^{2D}. If 1 spin is flipped there is 1 way it could be our given one. If two adjoining spins are flipped there are 2D ways one of them could be ours. If two non-adjoining spins are flipped then there are N^D-2D-1 ways one of them could be ours. So prob. $\downarrow=\frac{1}{Z}[\alpha^D+2D\alpha^{2D-1}+(N^D-2D-1)\alpha^{2D}+O(\alpha^{2D+1})]$

$=(1-N^D\alpha^D+O(\alpha^{D+1}))[\alpha^D+2D\alpha^{2D-1}+(N^D-2D-1)\alpha^{2D}+O(\alpha^{2D+1})]$

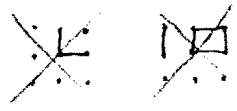
$=\alpha^D+2D\alpha^{2D-1}+(N^D-2D-1)\alpha^{2D}-N^D\alpha^{2D}+O(\alpha^{2D+1})$

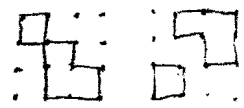
$=\alpha^D+2D\alpha^{2D-1}-(2D+1)\alpha^{2D}+O(\alpha^{2D+1})$

We see that the dependence on N has disappeared as it should.

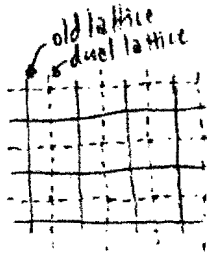
Let's now consider a high temperature expansion. In this

case $T \gg \frac{J}{k} \Rightarrow \beta J \ll 1$ so $\sqrt{\alpha} = \exp(-\beta J) \sim 1$. We are therefore led to writing our exponential of a sum as a product of exponentials: $Z = \sum_{\{\mu\}} \exp\left(-\beta \sum_{i,nn(i)} \frac{J}{4}(\mu_i - \mu_{nn(i)})^2\right) = \sum_{\{\mu\}} \prod_{i,nn(i)} \exp\left(-\beta \frac{J}{4}(\mu_i - \mu_{nn(i)})^2\right)$. Now notice that $\exp\left(-\beta \frac{J}{4}(\mu_i - \mu_{nn(i)})^2\right) = 1$ if $\mu_i = \mu_{nn(i)}$ and $= \exp(-\beta J) = \sqrt{\alpha}$ if $\mu_i \neq \mu_{nn(i)}$. So we can rewrite it (so as to get the μ's out of the exponential) as $\frac{1}{2}\{(1+\sqrt{\alpha}) + \mu_i \mu_{nn(i)}(1-\sqrt{\alpha})\} = \frac{1+\sqrt{\alpha}}{2}\left\{1 + \mu_i \mu_{nn(i)}\left(\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\right\}$ $= \frac{1+\sqrt{\alpha}}{2}\{1 + \mu_i \mu_{nn(i)} \gamma\}$ where $\gamma \equiv \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}$. Since $\sqrt{\alpha} \sim 1 \Rightarrow \gamma$ is a small parameter in terms of which we would like to expand Z. $Z = \left(\frac{1+\sqrt{\alpha}}{2}\right)^{DN^0}\left\{\sum_{\{\mu\}} \prod_{i,nn(i)}(1 + \mu_i \mu_{nn(i)}\gamma)\right\} = \left(\frac{1+\sqrt{\alpha}}{2}\right)^{DN^0}\left\{\sum_{\{\mu\}}\left(1 + \sum_{i,nn(i)} \mu_i \mu_{nn(i)}\gamma + \sum_{i,nn(i)} \sum_{j,nn(j) \neq i,nn(i)} \mu_i \mu_{nn(i)} \mu_j \mu_{nn(j)} \gamma^2 + \ldots\right)\right.$
Each pair i,nn(i) corresponds to one bond in our lattice. Each term in the γ^k coefficient is a product of spins along a path of k bonds in the lattice. We are summing over each of these spins being either 1 or -1. If any spin appears an odd number of times in a product then any contribution it makes when it is 1 will be cancelled when it is -1. The only paths that contribute are those with an even number of bonds at each lattice site, i.e. the closed paths one can traverse by not going over any bond more than once. Let's call the number of distinct such paths with k bonds in them: g(k). The product of spins in these paths is 1. Since there are 2^{N^0} states that we sum over, that is the coefficient of each path. The partition function becomes $Z = 2^{N^0}\left(\frac{1+\sqrt{\alpha}}{2}\right)^{DN^0} \sum_k g(k) \gamma^k$ with $g(0) = 1$.





In 2 dimensions there is an amazing connection between the two functions f(l) and g(k). In two dimensions the so-called dual lattice with spins at the centers of the squares of the old lattice is again a square lattice. If we associate



a flipped bond on the original lattice with an element of a path on the dual lattice, we see that the number of closed paths with k bonds is the same as the number of spin situations with k flipped bonds. Thus $f(\frac{1}{2}k)=g(k)$ for k even, =0 for k odd. So in two dimensions the partition function satisfies: $Z = 2^{N^2}\left(\frac{1+\sqrt{\alpha}}{2}\right)^{2N^2}\sum_{l=0}^{\infty} f(l)\gamma^{2l} = 2^{N^2}\left(\frac{1+\sqrt{\alpha}}{2}\right)^{2N^2}\sum_{l=0}^{\infty} f(l)\left(\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)^{2l}$

But we already saw: $Z=\sum_{l=0}^{\infty} f(l)\alpha^l$. If we define $\alpha' = \left(\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)^2$ then Z satisfies $Z(\alpha) = 2^{N^2}\left(\frac{1+\sqrt{\alpha}}{2}\right)^{2N^2} Z(\alpha')$. This expression relates the partition function at low temperatures $\alpha \sim 0$ to high temperatures $\alpha' \sim 1$. If there is to be only a single critical point, it must be when $\alpha=\alpha'$ (if not we would get at least two, one for some value of α and the other at the corresponding value of α') so $\alpha = \left(\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)^2 \Rightarrow \sqrt{\alpha}^2 + 2\sqrt{\alpha} - 1 = 0$ or $\sqrt{\alpha} = -1+\sqrt{2}$. And we see indeed $Z(-1+\sqrt{2}) = 2^{N^2}\left(\frac{1+-1+\sqrt{2}}{2}\right)^{2N^2} Z(-1+\sqrt{2})$

So if there is a critical point, it happens at $e^{\beta J} = \frac{1}{\sqrt{\alpha}} = \frac{1}{-1+\sqrt{2}} = 1+\sqrt{2}$

This is exactly the value for the critical point that the exact Onsager solution gives[47] (note that our J is twice that used in some expositions).

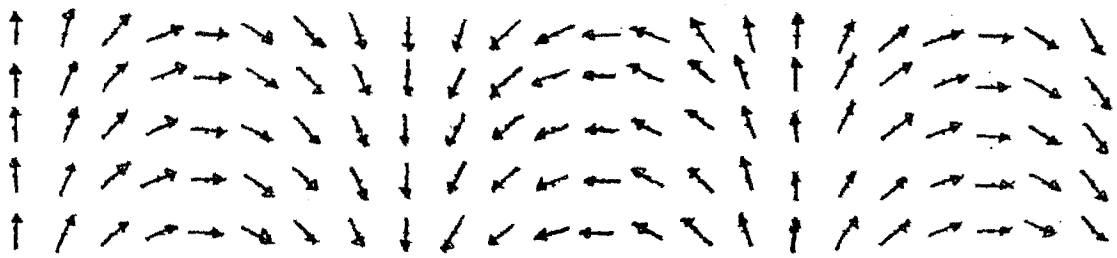
There is an interesting connection between the classical two dimensional Ising system we have been discussing and the quantum one-dimensional Ising system in a transverse magnetic field.[48] This is a 2^N dimensional Hilbert space (we could choose as a basis, vectors corresponding to each of the 2^N classical states). We then define a hamiltonian operator in the obvious way from our expression for the energy and an extra term for the external magnetic field. It turns out that there is a formal equivalence between this system and the classical

2-dimensional Ising system with one axis (which turns into the quantum time axis) taken to the continuum limit in a well-defined way. In the quantum system one can define a kink creation operator and an anti-kink creation operator that flip all spins up to a given point. The net kink number is a conserved quantity. One can understand the point corresponding to the phase transition in the 2-dimensional system in terms of kinks. At low temperatures, kink-anti-kink pairs form. As the temperature is raised, the pairs begin to separate until at some temperature the distance between elements of a pair is as large as the distance between the pairs and the system is disordered. We'll see a similar thing happening in the next model we consider.

The Ising model assumes that there is an anisotropy in the crystal that forces the spins to either align or anti-align with a given direction. A less restrictive model which allows rotation of the spin vector in a plane is known as the X-Y model. We will consider a 2-dimensional square lattice with a spin variable ϕ_i at each site i giving the angle of the spin direction. The energy of a given configuration is $E = -J\Sigma_{<ij>} \vec{S}_i \cdot \vec{S}_j = -J\sum_{<ij>} \cos(\phi_i - \phi_j)$ where $\Sigma_{<ij>}$ is a sum over nearest neighbors (each bond is counted twice).

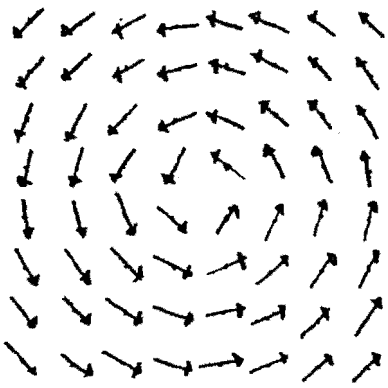
This model has no spontaneous magnetization for $T > 0$ because any energy at all can cause so-called spin waves. These are configurations where locally all spins point in almost the same direction but as we go in some direction

there is a slow net rotation of all the spins. The total energy of such a configuration is clearly lower for long wavelengths. In the limit of infinite wavelength, we just obtain the ground state of all spins pointing in the same direction. With arbitrarily little energy we can destroy global spin correlation.

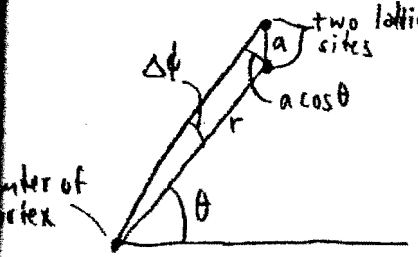


Thus it was thought that there could be no phase transition in the 2-dimensional X-Y model. Indeed, superposing spin waves on configurations where each spin is close to a given direction (what one might expect to be the appropriate low temperature approximation) gives no phase transition. Various models and numerical work, however, gave evidence that there was a transition of some sort (eg. susceptibility becoming infinite).

Just what was going on was not understood until the work of Kosterlitz and Thouless in 1972.[49] They realized that there were some other topologically stable configurations which they called vortices. If we integrate the change in spin direction around a loop, we can get non-trivial winding numbers: $\oint \vec{\nabla}\phi \cdot d\vec{l} = 2\pi q$, $q = 0, \pm 1, \pm 2, \ldots$. Call those with positive winding number vortices and those with negative, anti-vortices. We can work out the energy of such a configuration in a spin



system of radius R and lattice spacing a. Because far away from the vortex all nearest neighbors are close to one another in direction, it is reasonable to make the approximation: $\cos(\phi_i-\phi_j) \approx 1-\frac{1}{2}(\phi_i-\phi_j)^2$. Calling the energy of the ground state (all spins pointing in one direction) E_0 we see $E-E_0=\frac{1}{2}J\sum_{\langle ij\rangle}(\phi_i-\phi_j)^2$. We will first do the sum over all the vertical bonds (the symmetry of the situation implies that the sum over horizontal bonds will be the same). Labelling the sites by the polar coordinates r,θ we see that the difference in ϕ from one site to the vertically next one is: $\Delta\phi \simeq \frac{2\pi}{\left(\frac{2\pi r}{a\cos\theta}\right)} = \frac{a\cos\theta}{r}$. We can approximate our sum by an integral if we realize that each lattice site takes up an area a^2 (and remember to count bonds twice).



$(E-E_0)_{\text{vertical sum}} \simeq \frac{1}{2}J\int_{\text{area}} \frac{2\,d\vec{r}^2}{a^2}\left(\frac{a\cos\theta}{r}\right)^2 = J\int_{r=a}^{R}\int_{\theta=0}^{2\pi}\left(\frac{a\cos\theta}{r}\right)^2\frac{r\,d\theta\,dr}{a^2}$

$= J\int_{r=a}^{R}\frac{1}{r}dr\int_{\theta=0}^{2\pi}\cos^2\theta\,d\theta = \pi J\ln\left(\frac{R}{a}\right)$ So the total sum is: $E-E_0 \approx 2\pi J\ln\left(\frac{R}{a}\right)$

The energy of a vortex diverges as the system gets large. We might think, therefore, that free vortices are unimportant statistically. Recall, though, that the importance of a condition depends on the free energy which has an entropy term. Since our vortex can be at any of the $\sim\frac{R^2}{a^2}$ sites, the entropy of a single free vortex is $S=k\ln\left(\frac{R^2}{a^2}\right)$. The free energy is $F=E-TS=(2\pi J-2kT)\ln\left(\frac{R}{a}\right)$. When we reach the critical temperature $T_c=\frac{\pi J}{k}$, we see that the free energy

for a vortex is zero and we'll have lots of them populating the equilibrium state. When this vortex condensation occurs, spin-spin correlation functions die off exponentially since the vortices disorder spins over a small range.

We can understand physically how this transition occurs by reconsidering the low temperature regime. While we saw that free vortices can't occur, we can get some vortex-anti-vortex pairs. To work out the energy of some configuration of vortices it is easiest to work with continuum notation. A spin configuration is specified by a function $\phi(\vec{r})$. The energy of our lattice configuration should be proportional to $\iint d\vec{r}\,(\vec{\nabla}\phi)^2$. We can find the proportionality constant by considering $\vec{\nabla}\phi$ constant, say $\phi = xd$ so $\vec{\nabla}\phi = \hat{x}d$. The horizontal differences are da. With an area Na^2 with N spins, the energy is $JN(da)^2$. The integral gives : $C\iint d\vec{r}^2\,(\vec{\nabla}\phi)^2$ $= C\int d\vec{r}^2 (d)^2 = Cd^2Na^2$ so $C=J$. So $E=J\iint d\vec{r}\,(\vec{\nabla}\phi)^2$.

We know that pure vortex-anti-vortex configurations locally minimize the energy (fix their centers and they are stable). This is just a time independent field problem with Lagrangian $\mathcal{L} = J(\nabla\phi)^2 = J\,\partial_i\phi\partial_i\phi$. The Euler-Lagrange equation for extremitization is: $\partial_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 = \partial_i\left(2J\partial_i\phi\right) = 2J\nabla^2\phi$ So ϕ satisfies Laplaces equation $\nabla^2\phi = 0$ with singularities corresponding to vortices or anti-vortices. These aren't really singular points, they just correspond to loops around a single lattice spacing on the lattice. We've seen this kind of energy dependence. The vortices and anti-vortices

are like charged particles interacting via the Coulomb potential, modelled by ϕ. In two dimensions we know the solution to Laplace's equation gives logarithmic potentials (one can model a two dimensional system in three dimensions by making everything independent of one axis, i.e. consider interactions of infinitely long charged rods).

In the electromagnetic case we are used to the solution $\phi(x,y)=\sum_j q_j \ln\left(\sqrt{(x-x_j)^2+(y-y_j)^2}\right)$. Defining $z=x+iy$ and $z_j = x_j + iy_j$ we can rewrite this as $\sum_j q_j \mathrm{Re}[\ln(z-z_j)]$. This gives us radial dependence. What we want, though, is for our periodic variable ϕ to have angular dependence. We recall that since $\ln$ is an analytic function everywhere but at the origin, both the real and imaginary parts satisfy the two-dimensional Laplace's equation in the real and imaginary parts of their argument away from the origin. So $\phi(x,y)=\sum_j q_j \mathrm{Im}[\ln(z-z_j)]$ is a perfectly good solution. In fact, with a single charge at the origin, this is $\phi = \mathrm{Im}\ln(z) = \mathrm{Im}\ln(x+iy) = \mathrm{Im}\ln(re^{i\theta}) = \mathrm{Im}[\ln r + i\theta] = \theta$ just as we wanted if we define our charge $q_j = 1$ for a vortex and $=-1$ for an anti-vortex. Like our spins this definition is not single-valued when we go around a vortex. We would now like the energy of some pure vortex-anti-vortex configuration. We saw that it should be: $= J\iint d\vec{r}\,(\vec{\nabla}\phi)^2$.

With $\phi(x,y)=\sum_j q_j \mathrm{Im}\ln(z-z_i)$, $z = x+iy$ we find
$\partial_x\phi = \sum_j q_j \mathrm{Im}\left(\frac{1}{z-z_j}\right)$, $\partial_y\phi = \sum_j q_j \mathrm{Im}\left(\frac{i}{z-z_j}\right) = \sum_j q_j \mathrm{Re}\left(\frac{1}{z-z_j}\right)$
$S_0(\vec{\nabla}\phi)^2 = \left(\mathrm{Im}\left(\sum_j q_j \frac{1}{z-z_j}\right)\right)^2 + \left(\mathrm{Re}\left(\sum_j q_j \frac{1}{z-z_j}\right)\right)^2 = \left|\sum_j q_j \frac{1}{z-z_j}\right|^2$
But now notice that if we had used our usual potential:

$\phi'(x,y) = \sum_j q_j \, Re \ln(z - z_j)$ then $\partial_x \phi = \sum_j q_j \, Re\left(\frac{1}{z - z_j}\right)$

$\partial_y \phi' = \sum_j q_j \, Re\left(\frac{i}{z - z_j}\right) = -\sum_j q_j \, \mathrm{Im}\left(\frac{1}{z - z_j}\right)$

So $(\vec{\nabla}\phi')^2 = \left(Re\left(\sum_j q_j \frac{1}{z - z_j}\right)\right)^2 + \left(-\mathrm{Im}\left(\sum_j q_j \frac{1}{z - z_j}\right)\right)^2 = \left|\sum_j q_j \frac{1}{z - z_j}\right|^2 = (\vec{\nabla}\phi)^2$

So we can use the electrostatic ϕ' to calculate energies.

We can check that this gives the same energy for a single vortex as we calculated before. With a single unit charge particle at the origin we have: $\phi' = Re \ln(z) = \ln\left(\sqrt{x^2+y^2}\right) = \frac{1}{2}\ln(x^2+y^2)$

$\partial_x \phi' = \frac{x}{x^2+y^2}$ $\qquad \partial_y \phi' = \frac{y}{x^2+y^2}$ $\qquad (\vec{\nabla}\phi')^2 = \frac{1}{x^2+y^2} = \frac{1}{r^2}$

$E = J \iint d\vec{r}^2 \, (\vec{\nabla}\phi')^2 = J \int_{r=a}^{R} \int_{\theta=0}^{2\pi} \frac{1}{r^2} \, r \, d\theta \, dr = 2\pi J \left[\ln r\right]_a^R = 2\pi J \ln\left(\frac{R}{a}\right)$

as we calculated previously.

Let us now determine the energy in an infinite system for the case where there are the same number of vortices as anti-vortices. In this case we know things will settle out at infinity so that surface terms with $\vec{\nabla}\phi'$'s can be thrown away. We use the identity: $\vec{\nabla}\cdot(\phi\vec{\nabla}\phi) = (\vec{\nabla}\phi)^2 + \phi\nabla^2\phi$. So

$E = J \iint (\vec{\nabla}\phi')^2 d\vec{r} = J \iint \left(\vec{\nabla}\cdot(\phi'\vec{\nabla}\phi') - \phi'\nabla^2\phi'\right) d\vec{r}^2 = -J \iint \phi' \vec{\nabla}^2 \phi' \, d\vec{r}^2$

using the two dimensional Green's theorem: $\int_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{\partial A} M\,dx + N\,dy$

(easy to see with one forms) with $N = \phi' \frac{\partial}{\partial x}\phi'$ and $M = -\phi' \frac{\partial}{\partial y}\phi'$.

Our potential $\phi' = \sum_j q_j \frac{1}{2}\ln\left((x - x_j)^2 + (y - y_j)^2\right)$ satisfies $\nabla^2\phi' = 0$ everywhere but at the points (x_j, y_j). As in electromagnetism the energy can be thought of as comming from the action of the potential of all but one on that one summed over the particles. We'll take care of the self-energies later. Because $\nabla^2\phi'$ is non-vanishing only at a point we can take ϕ' out of the integral: $E = -J \iint \phi' \nabla^2 \phi' \, d\vec{r} =$

$= -J \sum_i \sum_{j \neq i} q_j \frac{1}{2} \ln\left((x_i - x_j)^2 + (y_i - y_j)^2\right) \iint_{\substack{\text{small region} \\ \text{about } (x_i, y_i)}} \nabla^2 \left(q_i \frac{1}{2} \ln\left((x - x_i)^2 + (y - y_i)^2\right)\right) dx\,dy$

We evaluate the double integral over a small circle by

coordinates to the origin and using Green's theorem again:

$$\iint = q_i \int_{\substack{\text{small circle}\\ \text{about origin}}} \left(-\frac{\partial}{\partial y}\left(\tfrac{1}{2}\ln(x^2+y^2)\right) dx + \frac{\partial}{\partial x}\left(\tfrac{1}{2}\ln(x^2+y^2)\right) dy \right) = q_i \int_{\text{circle}} \left(\frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \right)$$

Now use for dx, dy about a circle: $x = r\cos\theta$, $y = r\sin\theta \Rightarrow$

$dx = -r\sin\theta\, d\theta$, $dy = r\cos\theta\, d\theta$ $\quad \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$, $\frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$

$$\iint = q_i \int_{\theta=0}^{2\pi} \left(\left(\frac{-\sin\theta}{r}\right)(-r\sin\theta\, d\theta) + \left(\frac{\cos\theta}{r}\right)(r\cos\theta\, d\theta) \right) = q_i \int_{\theta=0}^{2\pi} d\theta = 2\pi q_i$$

So $E = -2\pi J \sum_{i\neq j} q_i q_j \ln|\vec{r}_i - \vec{r}_j|$.

In reality we cannot have vortices closer than a, so we can expect an error in the region when $|\vec{r}_i - \vec{r}_j| = a$. This and the self-energy term can be fixed by adding a piece to counteract the above when $|\vec{r}_i - \vec{r}_j| = a$ and a piece that gives the energy needed for creation of the vortices. If the energy to create a vortex-anti-vortex pair, each with vorticity of magnitude q, is $2\mu q^2$ (μ acts like a chemical potential) then the energy is: $E = -2\pi J \sum_{i\neq j} q_i q_j \ln\left|\frac{\vec{r}_i - \vec{r}_j}{a}\right| + \mu \sum_i q_i^2$. On top of this is the energy of spin waves and other excitations with no vorticity.

At low temperatures the system will be dominated by vortex-anti-vortex pairs that are closely bound and far from other pairs. The energy of a pair (with vorticity q and -q) separated by r is: $E_{pair} = 4\pi J q^2 \ln\left(\frac{r}{a}\right) + 2\mu q^2$. We know that the probability for a given r is $\propto e^{-\beta(\text{energy for that } r)}$ so we may find the mean square separation of the pair:

$$\langle r^2 \rangle = \frac{\int_{r=a}^{\infty}\int_{\theta=0}^{2\pi} r^2 \exp\left(-\beta 4\pi J q^2 \ln\left(\frac{r}{a}\right) + 2\mu q^2\right) r\, d\theta\, dr}{\int_{r=a}^{\infty}\int_{\theta=0}^{2\pi} \exp\left(-\beta 4\pi J q^2 \ln\left(\frac{r}{a}\right) + 2\mu q^2\right) r\, d\theta\, dr}$$

$$= \frac{\int_{r=a}^{\infty} r^3 \exp\left(-\beta 4\pi J q^2 \ln\left(\frac{r}{a}\right)\right) dr}{\int_{r=a}^{\infty} r \exp\left(-\beta 4\pi J q^2 \ln\left(\frac{r}{a}\right)\right) dr}$$

The numerator is $= \frac{1}{a^{-\beta 4\pi J q^2}} \int_{r=a}^{\infty} r^{3-\beta 4\pi J q^2} dr = a^{\beta 4\pi J q^2} \left[\frac{1}{4-\beta 4\pi J q^2} r^{4-\beta 4\pi J q^2} \right]_a^{\infty} = \frac{-1}{4-\beta 4\pi J q^2} a^4$ if $\beta 4\pi J$

The denominator is $= a^{\beta 4\pi J q^2} \left[\frac{1}{2-\beta 4\pi J q^2} r^{2-\beta 4\pi J q^2} \right]_{r=a}^{\infty} = \frac{-1}{2-\beta 4\pi J q^2} a^2$

So $\langle r^2 \rangle = a^2 \frac{\beta 2\pi J q^2 - 1}{\beta 2\pi J q^2 - 2}$

If the mean separation between pairs is d then we are likely to find one pair in an area d^2. Thus $\frac{1}{d^2}$ is the probability per unit area of finding a pair. The probability per unit area is also $= \left(\sum_{\substack{\text{all pairs} \\ \text{in area}}} \exp(-\beta \times \text{energy of pair})\right) \Big/ \left(\text{Area} \times \left(\text{Same} \sum + \exp(-\beta \times \text{energy of no pair})\right)\right)$ The way we have defined things, the energy with no pairs is zero. So if $\sum \ll 1$ then $\text{Probability/Area} \approx \left(\sum_{\text{all pairs in area}} \exp(-\beta \text{energy of pair})\right) / \text{area}$ $= \sum_{i,j \text{ in area}} \exp\left(-2\beta\mu q^2 - \beta 4\pi J q^2 \ln\left(\frac{r}{a}\right)\right) / \text{Area}$. We may turn this into a double integral if we divide by a^4. One of the integrals approximately gives the area and cancels with the denominator. The rest is: $\frac{1}{d^2} = e^{-2\beta\mu q^2} \int_{r=a}^{\text{radius } R} 2\pi \left(\frac{r}{a}\right)^{-\beta 4\pi J q^2} r \frac{dr}{a^4} =$ $= 2\pi e^{-2\beta\mu q^2} a^{\beta 4\pi J q^2} \frac{1}{a^4} \left[\frac{1}{2 - \beta 4\pi J q^2} r^{2-\beta 4\pi J q^2}\right]_a^R \approx \frac{\pi}{a^2} e^{-2\beta\mu q^2} \frac{1}{\beta 2\pi J q^2 - 1}$ for $\beta > \frac{1}{4\pi J q^2}$ Thus $\left\langle \left(\frac{r}{d}\right)^2 \right\rangle \approx \frac{\pi e^{-2\beta\mu q^2}}{\beta 2\pi J q^2 - 2}$. We see that when $\beta 2\pi J q^2 = 2$ or equivalently $T_c = \frac{\pi J q^2}{2}$, the separation between components of pairs gets much larger than the distance between pairs, i.e. the vortices become free. This is exactly the result we obtained before for T_c, with q=1.

In reality, as a pair gets more and more widely separated, the presence of the other pairs relaxes the field and lowers the energy. It is possible to model their effect by an effective "dialectric constant" as we would do in the case of a gas of electric dipoles. The only effect, though, is to renormalize the critical temperature and qualitatively our description is correct.

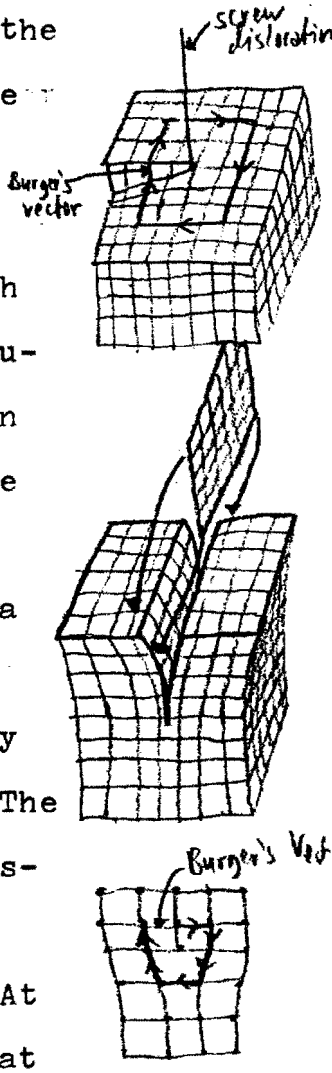
Let's summarize the Kosterlitz-Thouless picture.[50] At low temperatures there is no net vorticity and so there is local order. The state is dominated by spin waves and bound

vortex-anti-vortex pairs. The spin waves contribute nothing to the phase transition. As we raise the temperature, more and more pairs form and the components of each pair get farther apart. At some critical temperature, the distance between components of a pair get much farther apart than the distance between pairs. At this point, the vortices are essentially free. Large regions can have a net vorticity and we have a disordered phase.

This picture has had great impact. It applies directly to the case of thin super-fluid helium films.[51] In the Ginzburg-Landau model there is a single complex order parameter $\psi(r) = |\psi| \exp(i\phi)$. The phase serves as the analog of our spins ϕ (in the continuum version) since the superfluid velocity is $\vec{v} = (\frac{\hbar}{m})\vec{\nabla}\phi$ and the energy has a term with the integral over $\vec{v}^2 \propto (\vec{\nabla}\phi)^2$. The vortices in this case are physical vortices in the helium film. They affect such things as the flow. Experimental confirmation of the results expected by the Kosterlitz- Thouless theory have been reported.[52] With some complications, the theory applies to "dirty" superconducting thin films as well.[53]

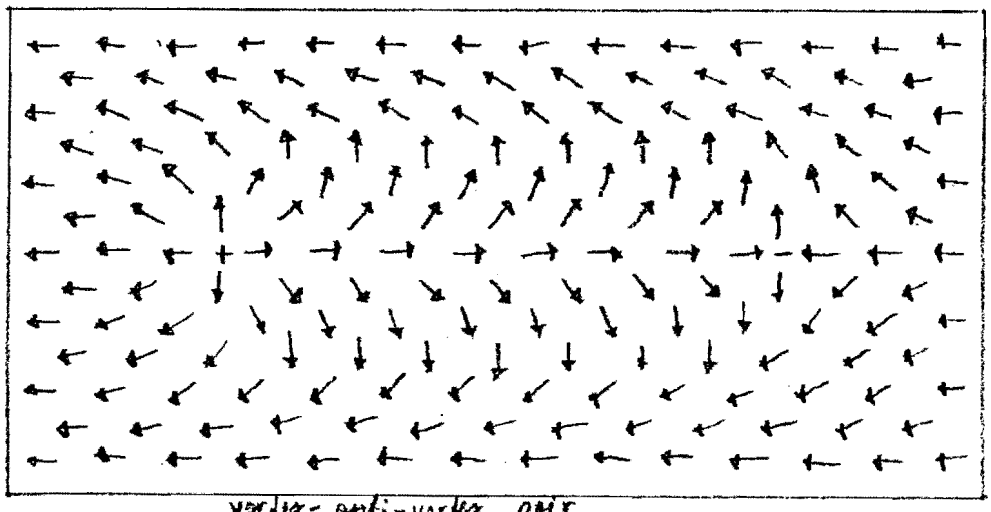
It is also applicable in an interesting way to the understanding of solid-liquid transitions.[51] In a crystal there are large regions (called grains) within which there is local order (say atoms in a cubic lattice). In three dimensions, these grains are permeated by a web of lines known as crystal dislocations. These occur because, even though locally everything tries to be on a cubic lattice, sometimes it is topologically

impossible, due to the way things have connected in one region to continue the crystal structure to a perfect cubic lattice everywhere. There are two kinds of line dislocations in three-dimensions and they are characterized by what is known as the Burger's vector. This is the vector needed to complete a traversal of the lattice around the dislocation line that would close in a perfect crystal (note the similarity to the idea of curvature!). The first kind of dislocation is the screw dislocation. When you make a loop around the line, you end up shifted up a plane of atoms. These are energetically popular configurations for crystals since starting a new face in crystal growth is hard, whereas continuing growth on a ragged edge is easy. If a growth gets started growing helically it has no trouble continuing. The Burger's vector is parallel to the dislocation line in this case. The other kind arises when atoms are in perfect planes perpendicular to the dislocation, but within a plane there are more atoms accross the top of a square around the dislocation than the bottom. You can imagine taking a perfect crystal, making a slice halfway through it, and inserting a half plane of extra atoms. The Burger's vector in this case is perpendicular to the dislocation line. The dislocations can move about in the crystal, the higher the temperature, the more easily. At a critical temperature, the dislocations are so free that the response to a small external shear stress changes from

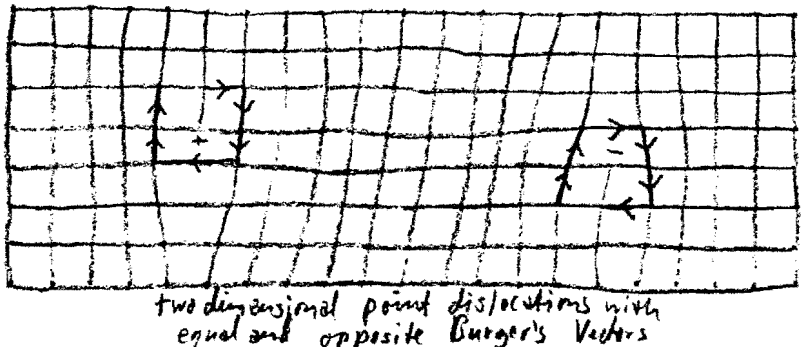


elastic to fluid (the dislocations can move to the stressed area).

In two-dimensions we have point dislocations. It turns out that the energy of a single dislocation is logarithmically dependent on the size of the system. In addition, the energy of pairs of dislocations goes logarithmically with the distance between them. All of our previous results hold for two dimensional crystals. A single dislocation acts like a free vortex. A pair of dislocations with equal and opposite Burger's vectors acts like a vortex-anti-vortex pair. We can thus understand the solid-liquid transition in terms of pairs of dislocations with opposite Burger's vectors forming at low temperatures. As the temperature is raised, they eventually become free, disordering the crystal and allowing fluid response to shear stress, i.e. turning the solid into a liquid.



51.



Conclusion

What can we conclude from all this? The Kosterlitz-Thouless vortices have been a brilliant success but use only the simplest of topological ideas. The field theory lumps have as yet only appeared classically and have at present no observational significance. The theories, however, have a certain richness that suggests that similar ideas may bear fruit in other places. There are some very general features of theories that are easy to understand when looked at from a topological standpoint (eg. the solitons in sine-gordan theory or the stability of systems with feedback). There is the possibility, for example, that more conservation laws will be found to have a topological character, thus further illuminating their mechanism. It appears that topology gives us a way of creating new levels of objects (eg. we can consider vortex interactions independently of the underlying spin interactions). Topology appears to be a rich source of concepts. In particular, its very powerful algebraic analysis of seemingly non-algebraic phenomena appears to bear a close resemblance to the present physical theory's analysis of physical phenomena. As has happened so many times in the past, perhaps the richness of this language will allow us to express new kinds of truths of the universe. It is my feeling that in some form, topological ideas in physics are here to stay.

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