Mathematical Biosciences

an international journal

Volume 68 Number 2 April 1984

Elsevier

BOOK REVIEW

C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, Springer Verlag, New York, 1982, 269 pp., Softback, $19.80

Around the turn of the century Poincaré reported in his work on celestial mechanics that very simple systems of differential equations may have very complex solutions and he initiated the geometric/topological approach to the qualitative description of the equations. In the ensuing years, mathematicians have developed this approach quite extensively. Two very good references are Guckenheimer and Holmes [1] and Arnold [2]. These ideas did not have much of an impact on the natural science community until about ten years ago, when the meteorologist E. N. Lorenz's 1963 paper [3] became widely known.

In Lorenz's paper, the three dimensional system of differential equations, which have become known as the Lorenz equations, were introduced:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = xy - bz$$

where σ, r, and b are three real positive parameters. These equations represent a three-mode truncation of the fluid equations that describe convection in a layer (like the atmosphere) heated from below. Lorenz found that for the parameter values $\sigma = 10$, $b = 8/3$, and $r = 28$, the solution curves in x, y, and z space loop around two equilibrium points, alternating from one to the other in a seemingly random way. Furthermore, very small changes in the initial conditions lead to entirely different sequences of loopings. Lorenz showed that from this very simple deterministic system, one may obtain effectively unpredictable behavior. Lorenz's hope is that this kind of behavior is responsible for the notorious limitations of weather forecasting. Since 1970, many other simple systems of equations and experimental physical systems with chaotic dynamics have been found. Many of these systems have been extensively studied numerically and, for some, fairly complete theoretical understanding has been obtained.

This book attempts to tie together the enormous amount of research on the Lorenz equations in limited parameter regimes and show how this research fits together. The resulting picture is staggering in its complexity. Nearly every

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known chaotic behavior occurs for some parameter values in the Lorenz system and Sparrow describes them admirably. The book is written very carefully and Sparrow at each point explains what is known rigorously, what is known beyond a reasonable doubt from numerical data, and what is merely strongly suggested by numerical data.

In the first six chapters he takes us through the ever growing complexity as r increases from zero to infinity, while $\sigma = 10$ and $b = 8/3$. The story unfolds like a detective novel with each new bifurcation beginning with some tantalizing clues and confusing mysteries about its behavior. The analysis proceeds and the intricate structure necessary for explanation is unfolded. As r increases, a stable fixed point becomes unstable and gives birth to two new stable fixed points. These are then eaten by unstable orbits in a subcritical Hopf bifurcation. Where did the orbits come from? While we weren't looking there was a homoclinic explosion that created an incredibly complex fractal set with an infinite number of periodic and random orbits. This set becomes a strange attractor and, meanwhile, an infinite number of further homoclinic explosions are going off. Things begin hooking to form Smale horeshoes and stable periodic orbits. All these new orbits begin disappearing through infinite period doubling cascades. At several places, the ghosts of attracting sets make their presence known through intermittent behavior. All these phenomena and more are explained as they are encountered.

Chapters seven and eight begin exploring other values of σ and b. Amazingly, the complexity goes up again and it appears that there is still much to be discovered. The last chapter discusses extensions of the Lorenz system and other approaches to its study. Eleven appendices give mathematical definitions and results that are referred to in several places throughout the book.

The revelation of this amount of complexity, not only in the flow for a given set of parameters, but also in the variety of behaviors and the intricacy of their interconnection, in a system of O.D.E.'s with only quadratic nonlinearities should be sobering to any user of differential equations. The variety of techniques employed in the elucidation of this system's behavior should be useful even to workers to whom the Lorenz equations are not of direct interest. The book does not have many formal prerequisites, but a certain familiarity with the techniques (as described in [1] or [2]) may be necessary for full understanding of details, especially in the later sections.

REFERENCES

1 J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*, Springer Verlag, New York, 1983.

2 V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer Verlag, New York, 1983.

3 E. N. Lorenz, Deterministic non-periodic flows, *J. Atmos. Sci.* 20:130–141 (1963).

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