TEN IDEAS FROM NONLINEAR DYNAMICS

by Stephen Omohundro
at Thinking Machines Corp.
on Oct. 31, 1984

1. MANIFOLDS
2. DEFECTS
3. STRUCTURAL STABILITY
4. FOUNDATIONS OF DYNAMICS
5. BIFURCATION THEORY
6. CHAOS
7. HAMILTONIAN SYSTEMS
8. RENORMALIZATION
9. CURVATURE
10. ADIABATIC INVARIANCE

WHERE DOES NONLINEAR DYNAMICS ARISE IN ARTIFICIAL INTELLIGENCE?

NONLINEAR DYNAMICS

↕

NONELEPHANT ZOOLOGY

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in interfacing with the world:

MACHINE MANIPULATING WORLD:

- Robotics (dynamics of arms, etc.)
- Speech + Sound Synthesis
- Graphic Display (Hamiltonian ray tracing)

MACHINE SENSING WORLD:

- Vision (early processing)
- Hearing
- Touch, Chemical sensing, electrodynamic, etc.

PROCESSING ALGORITHMS:

Relaxation procedures in vision

Optimization by Simulated Annealing
(chip + board layout - Kirkpatrick, Gelatt, Vecchi)

DENDRAL - the test in generate & test

MODELING GROSS FEATURES OF COMPLEX SYSTEMS

(eg. throughput of highly parallel computer)

Statistical Mechanics

(.. Montroll)

Sears Roebuck Catalogs - log normal price distribution w/ constant dispersion for 75 years

Freeway Traffic - laws for flow instability

Language - probability of nth most popular word is $\frac{1}{n}$ to very good approximation $1 \leq n \leq 10{,}000$

Maximum Entropy

ENGINEERING OF COMPONENTS

TECHNIQUES OF DYNAMICS AS THINKING PARADIGMS

(which a machine should be able to use)

FUNCTIONS AND MAPPINGS AS OBJECTS ⟷ PROCEDURES AS ARGUMENTS + RETURNED VALUES

FUNCTORIAL OPERATIONS (eg. pullback) ACT ON MANY TYPES OF OBJECT ⟷ DATA-DRIVEN PROGRAMMING

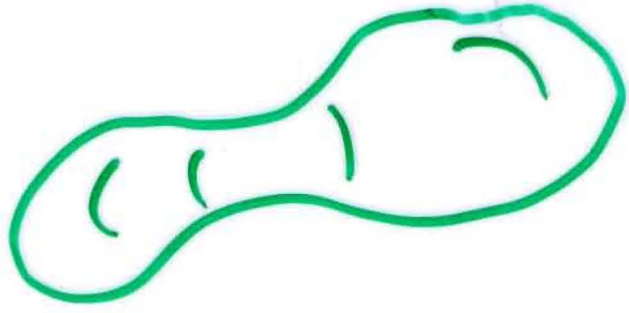
COORDINATE FREE CONCEPTS UNIVERSALITY ⟷ DATA ABSTRACTION

KEY TO ALL OF MATHEMATICS ⟷ DELAYED EVALUATION

POINCARÈ:

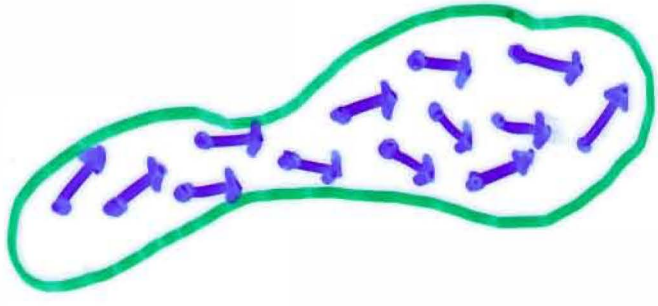
To understand one instance of something, consider it in relation to nearby instances.

STATE SPACE:



one point for every state of the system

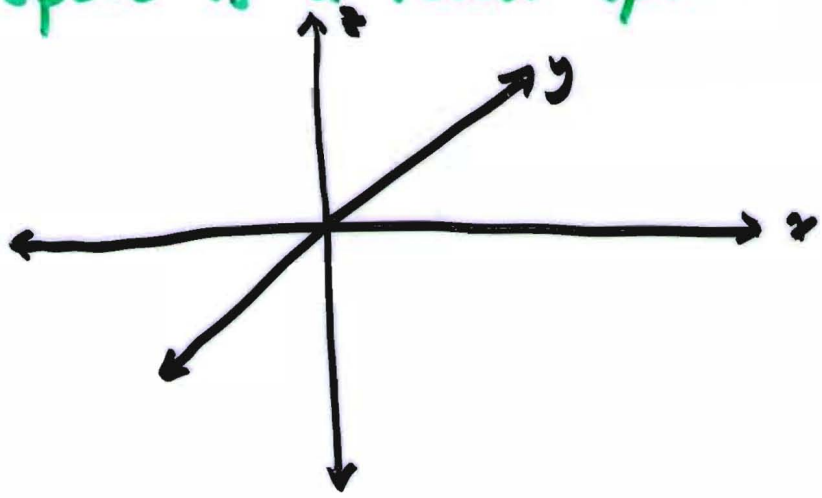
DYNAMICAL SYSTEM:



state space + dynamical vector field

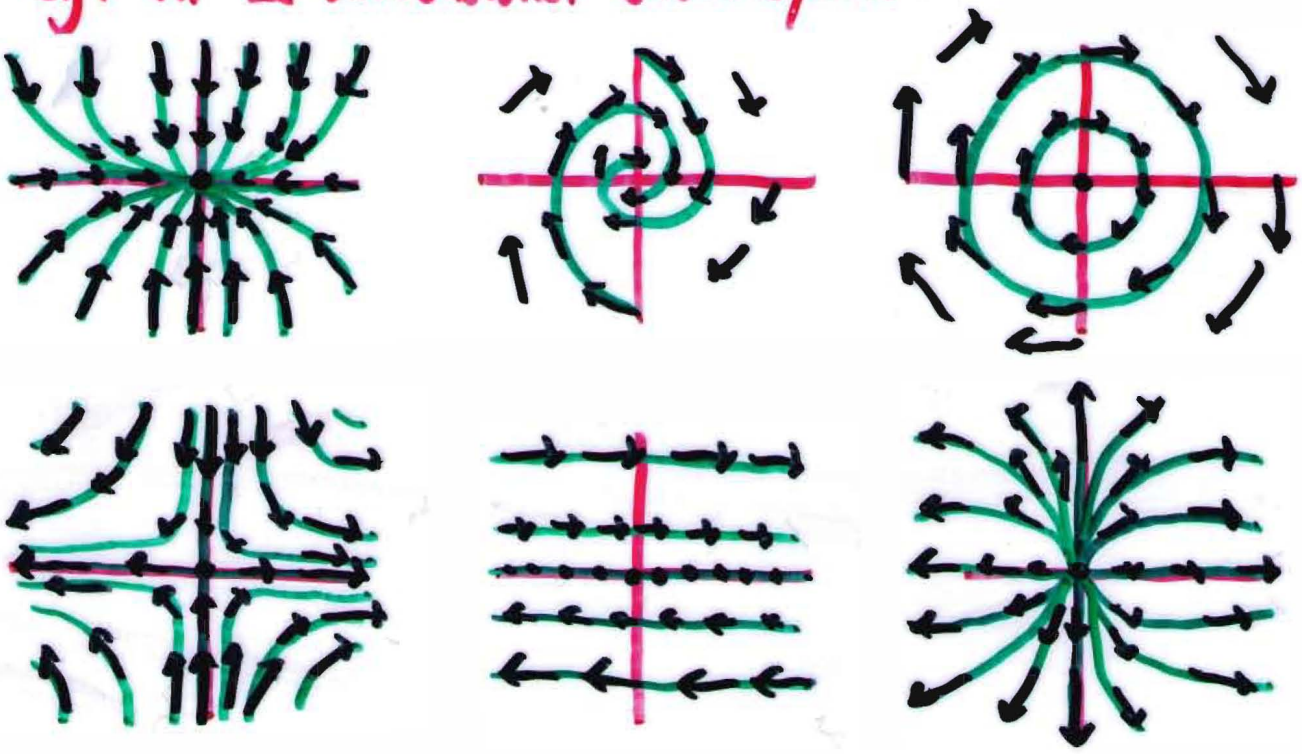
LINEAR DYNAMICAL SYSTEM:

State space is a vector space:



Dynamics is a linear vector field:

eg. in 2-dimensional state spaces:



A SYSTEM IS NON-LINEAR IF ITS DYNAMICAL VECTOR FIELD IS NON-LINEAR OR ITS STATE SPACE IS NOT A LINEAR SPACE.

STATE SPACES ARE TYPICALLY

MANIFOLDS.

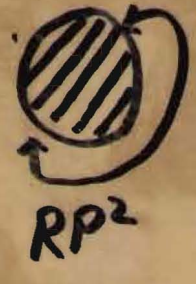
(H. Weyl ~ 1920) - a space that locally looks like Euclidean space (i.e. $\exists$ local coordinate patches) but globally may not.

eg Spheres:

Tori:

Multihandled2tori:

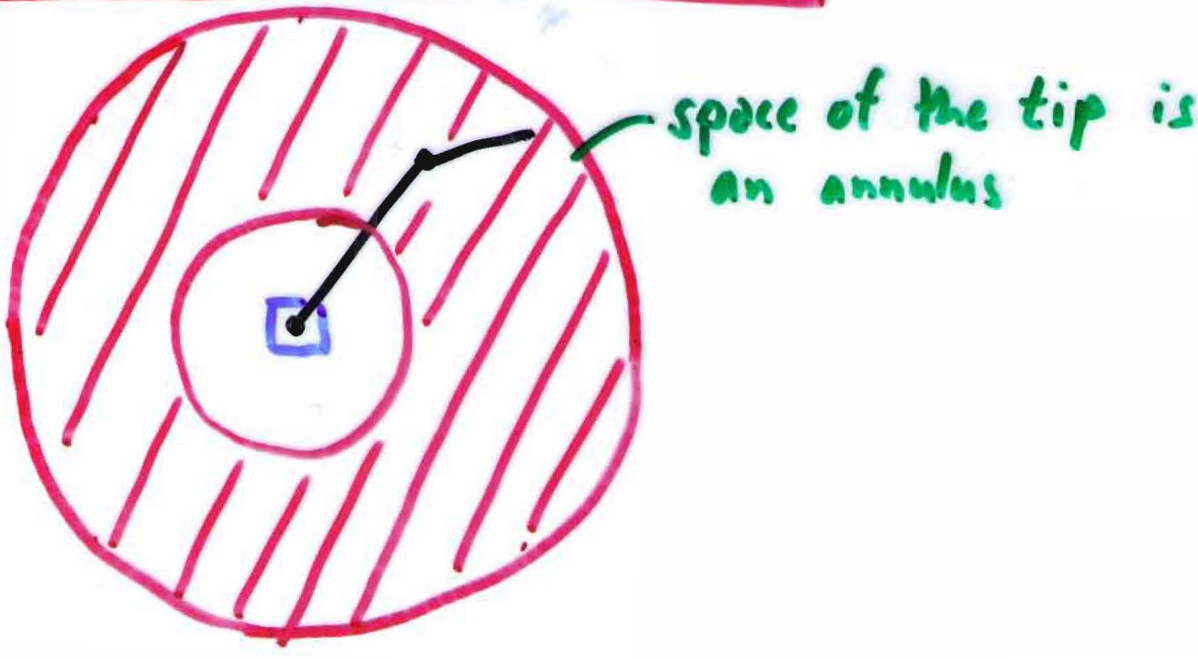
Projective spaces:

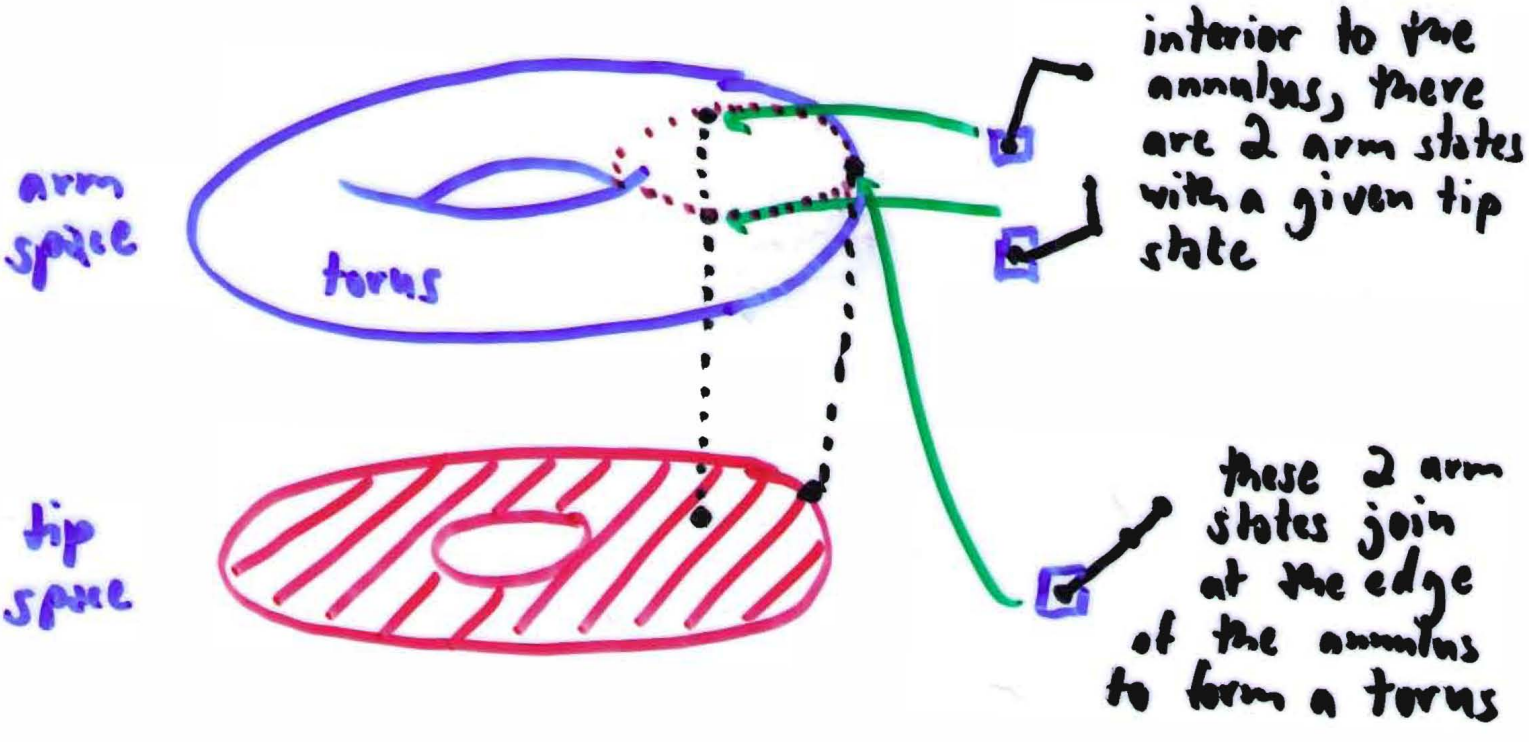


Disc with opposite edges identified

- like asteroids game but with extra flip

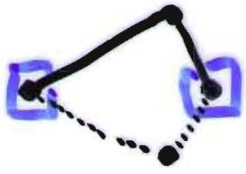
2-dim, 2-link robot arm





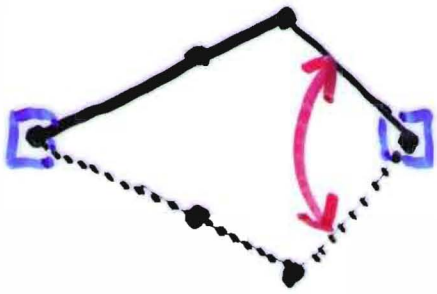
2 links in 2-dims, both ends fixed

(think of arm state with given end effector state)

Arm	Configuration Space
	2 points
	1 point
	1 point

3-links in 2-dims, both ends fixed

3rd link is restricted to angle - its state is a point on a closed interval



Hold 3rd link fixed → Get the 2 link problem we already understand:

3 in interior → 1 & 2 have 2 states

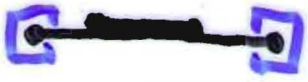
3 at endpoints → 1 & 2 have a single state

SO THE STATE SPACE OF THE ARM IS A CIRCLE.

3 links, 2-dims, ends fixed

Now push the ends closer together:

When close enough – only get one state when 3 is horizontal:



Configuration space:

middle of circle collapses

Pushing the ends even further together:

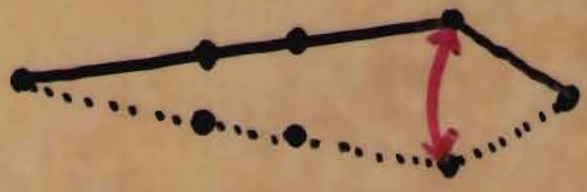
3 Can't even get to the middle

Configuration space:

2 circles

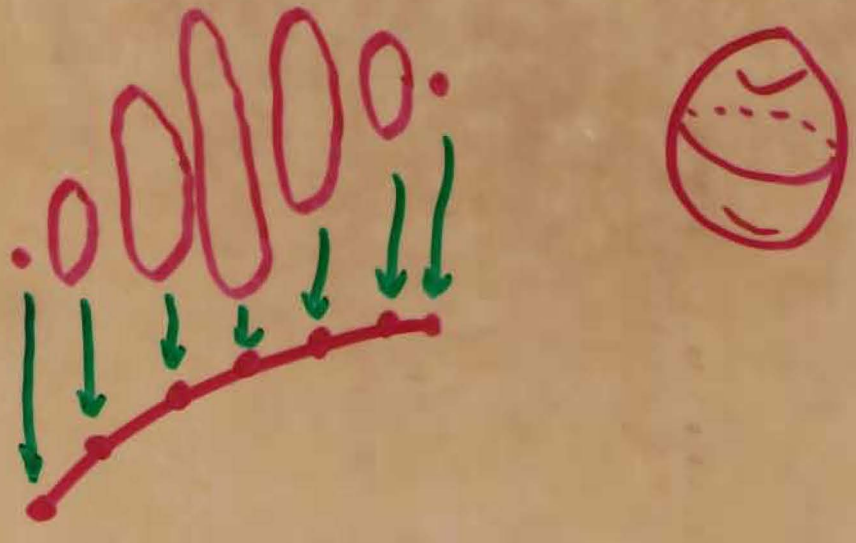
4 links, 2-dims, ends fixed.

Again state of last link is an interval:



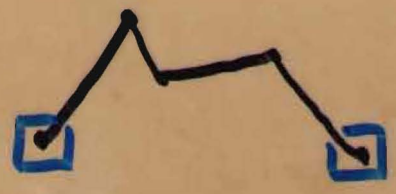
State of 1st 3 links is a point at ends, in middle we saw it was a circle:

So: state space is a 2-sphere.



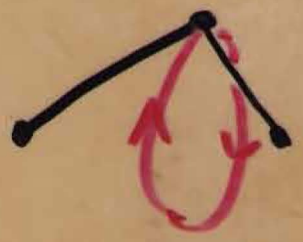
4 links, 2-dim, ends fixed

Push them closer to force the 3-link circle to break in the middle:



So state space of arm is a 2-torus

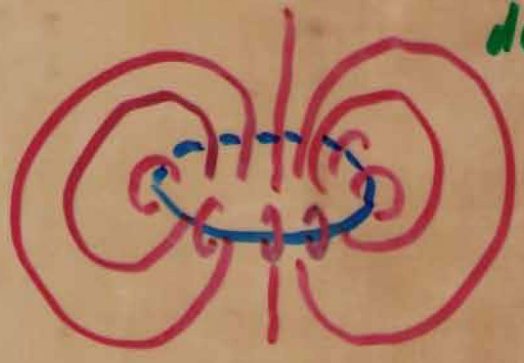
2 links, 3-dims, ball joints, ends fixed



state space is a
Circle $= S^1$

3 links, 3-dims, ball joints ends fixed
(Feeling manifolds in your bones)

3rd joint lies in a disk
Other 2 form circles which degenerate at edge of disk



state space is a
3-sphere $= S^3$

Rotate links about line joining ends to fill up S^3 with circles – Hopf fibration

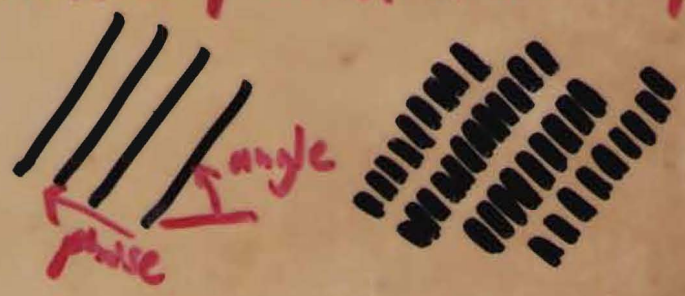
Each circle links every other one
$\Rightarrow$ constraints on deformation of arm.

eg Twiddling thumbs with string tied is Hopf fibration

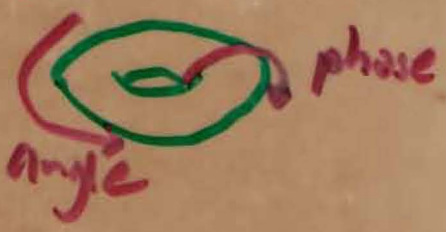
rotor in plane – nematic liquid crystal in plane

State space is circle:

periodic parallel lines in plane – smectic



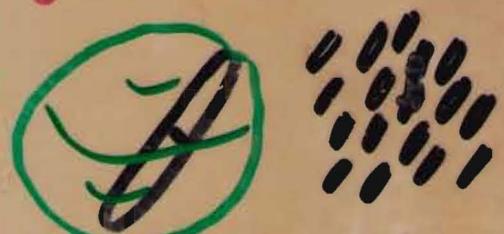
State space is torus



Arrow in 3-d fixed at 1-pt.

State space is 2 sphere S^2

Cigar in space – fixed at center 3-d nematic

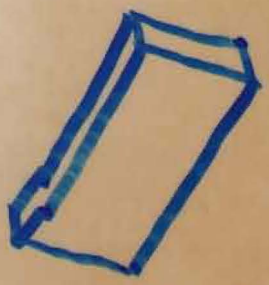


State space is 2 sphere with diametrically opposite pts. identified $= RP^2$ 2-d projective plane

can think of as disk with opposite points on edge identified:

by looking at upper hemisphere (like "asteroids" but with a flip)

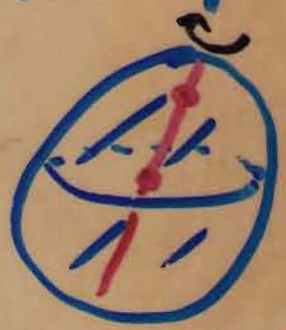
Rigid body in 3-d (eg. direction of end effector)



Rotation Group

RP^3 – Real projective plane of 3-dims

Choose reference configuration – every other one is gotten by rotating about some axis. So state space is ball:

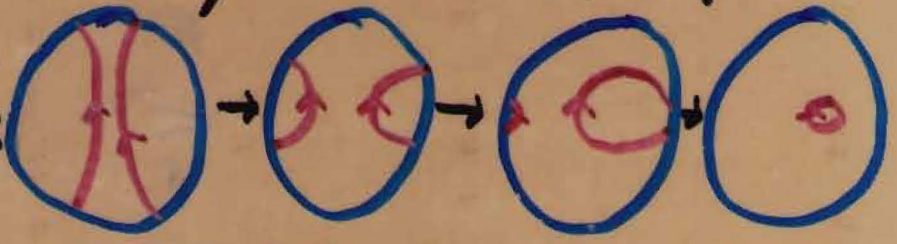


Center is reference state

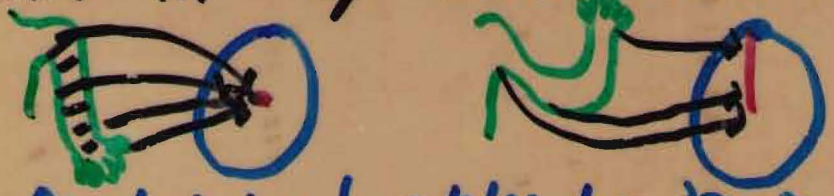
Any other pt. is rotation about axis between it and center by angle = distance to center

Rotation by π & $-\pi$ give same state so ball has radius π and opposite points on surface are identified. $\Rightarrow = \mathbb{R}P^3$

Has loops that are not smoothly contractible to a point.

2π:  but 4π: 

Map arm into RP^3 by rotation each little slice undergoes:



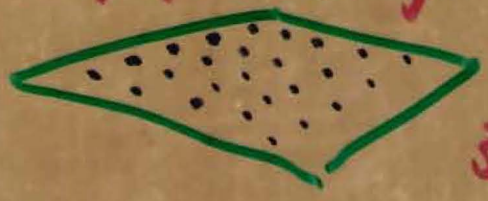
If shoulder fixed & hand rotates by $2\pi \Rightarrow$ arm defines non-contractible loop.

If hand rotates by 4π, can smoothly deform arm back to original state keeping hand fixed.

Sine wave on a line - 1-dim crystal

state space is a circle

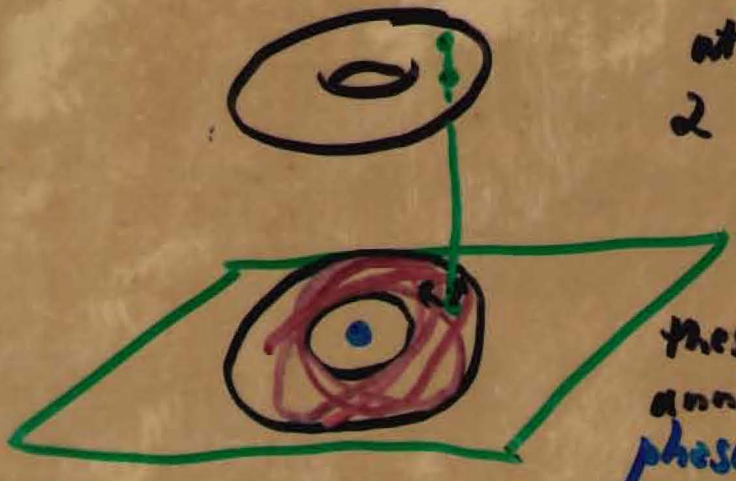
lattice in plane of given orientation - 2-d crystal



state space T^2 torus

3-d crystal get T^3

Planar particle in rotationally symmetric potential:



at given radius there are 2 velocities consistent w/ energy & angular momentum (inward & outward)

these degenerate at edge of annulus → motion in the phase space is on a 2 torus

Lissajou figures:

at each point there are 4 possible directions

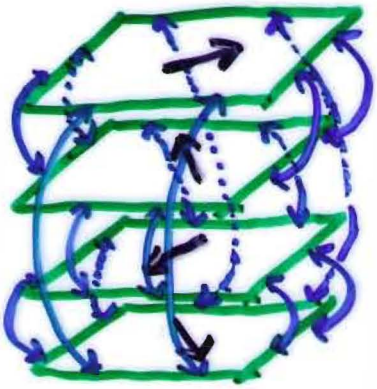
→ get 4 sheets glued together at edges

→ 2 torus

The corresponding classical systems are "billiard tables". To

In these 2-d examples the flow is on surfaces in phase space.

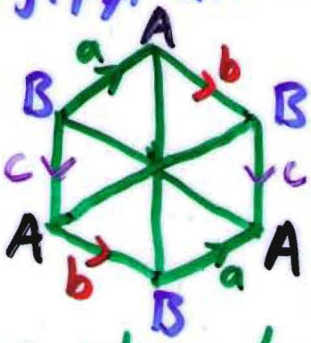
Eg. Consider  . Starting w/ momentum ↗ can only get 3 others: ↖, ↙, ↘. So in phase space motion is on 4 sheets:



Bouncing off an edge means jumping sheets.

Hooking up the appropriate sheets we get a torus! The motion looks integrable - indeed $|p_x|, |p_y|$ are conserved.

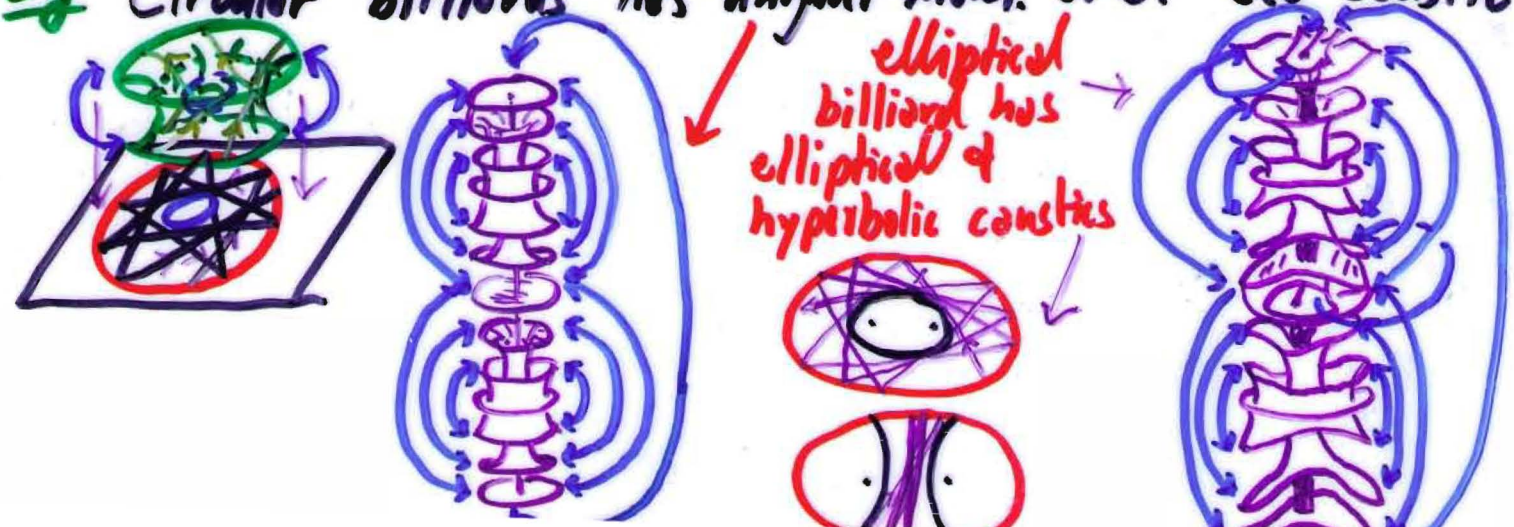
Eg. Consider . Get 6-sheets →



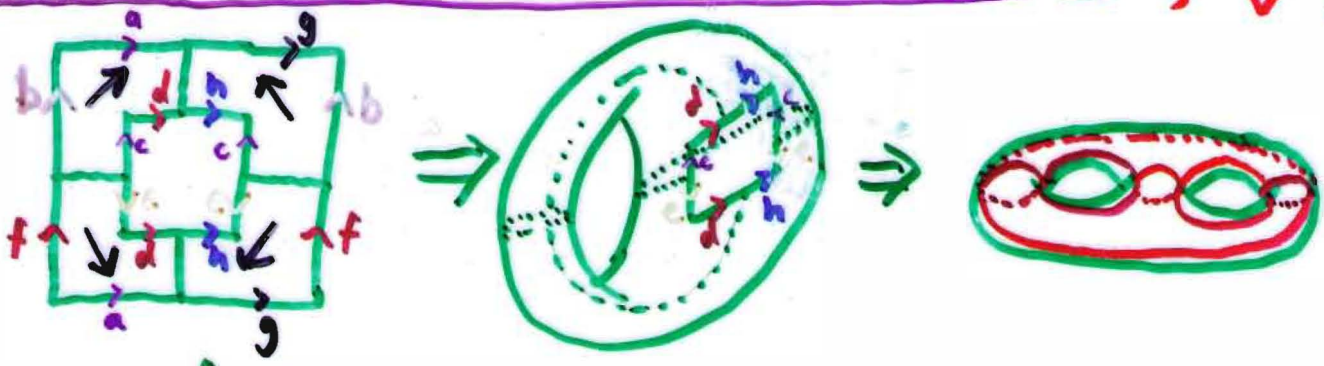
Again get torus  = integrable.

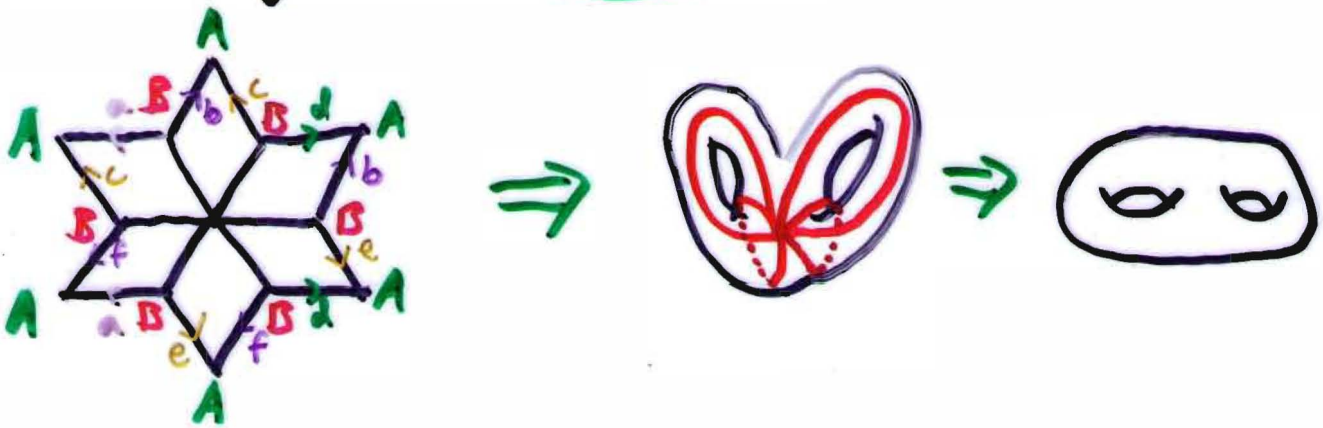
Can show anything which tiles the plane by reflection yields a torus: So  is integrable.

Eg. Circular billiards has angular mom. cons. - see constic



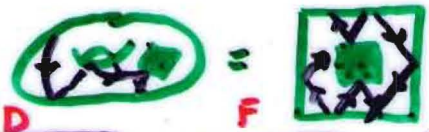
WHAT HAPPENS WHEN WE TRY: ⌐, ◇, ◗ ?

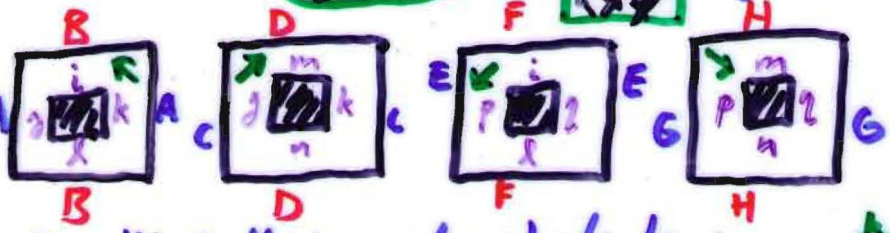




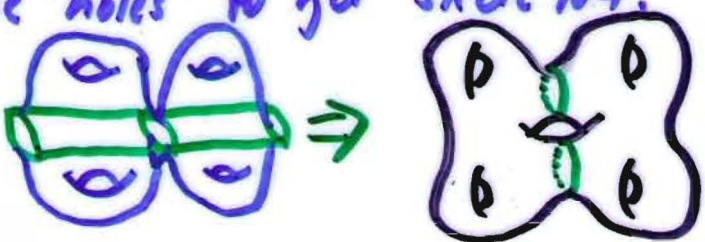
These are two-holed tori. Arnold's theorem doesn't apply since the "corners" are singularities (even though p_x^2, p_y^2 are conserved in ⌐ eg.)

Systems with flow on N-dim surfaces are called <u>pseudo-integrable</u>.

). Past in toral billiard:  get 4 sheets in

hase space: 

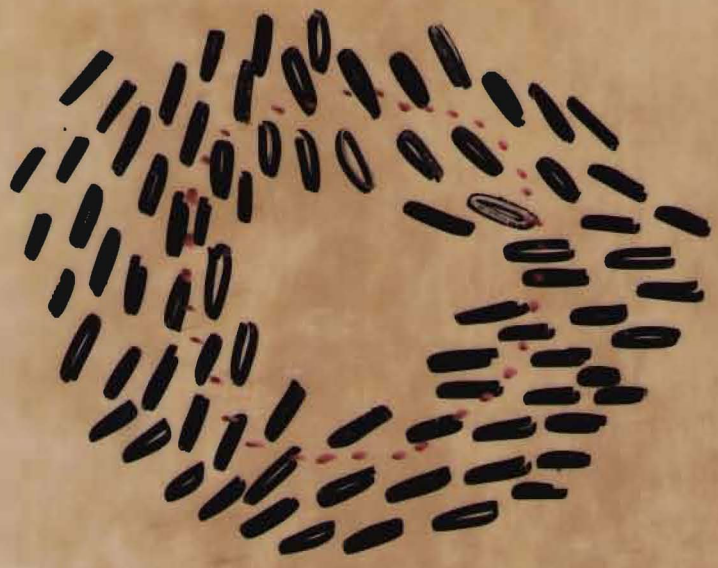
First connect the "holes" to get skeleton:

ew" on the tori: 

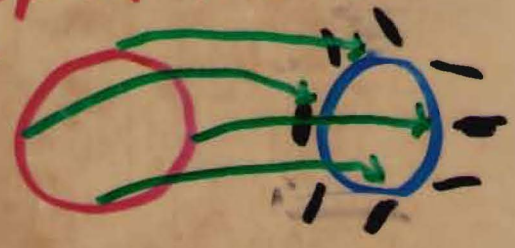
A 5-holed torus!!

Topological Defects

Spill some nematic liquid crystal on the plane:



As we describe a loop in the plane the local state describes a loop in the state space



because this loop is non-contractible, there is no way to continuously fill in the hole.

⇒ get topological pocket of higher energy hard to get rid of since must cut a lot of bonds ⇒ local minimum of energy

defects move around & interact like particles.

In plane, defects are labeled by non-contractible loops in state space.

Push 2 together, get new one. Its loop is gotten by composing 2 original loops.

⇒ Group structure = "First Homotopy Group" $\pi_1(S)$

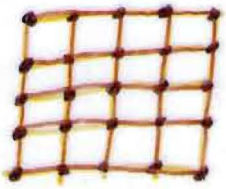
eg

A.) "Defect" ideas in physics

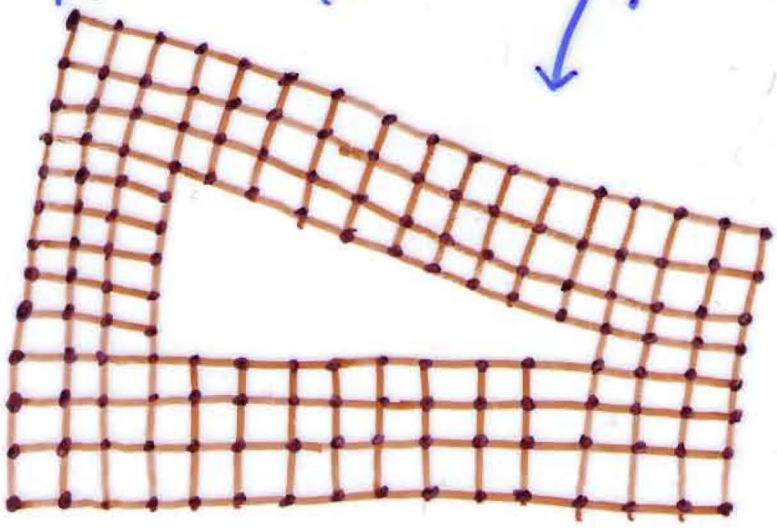
Global structure can prevent the attainment of a desirable local structure everywhere in space. The imperfections can move around but cannot be destroyed for topological reasons.

Eg. 2-dimensional crystal

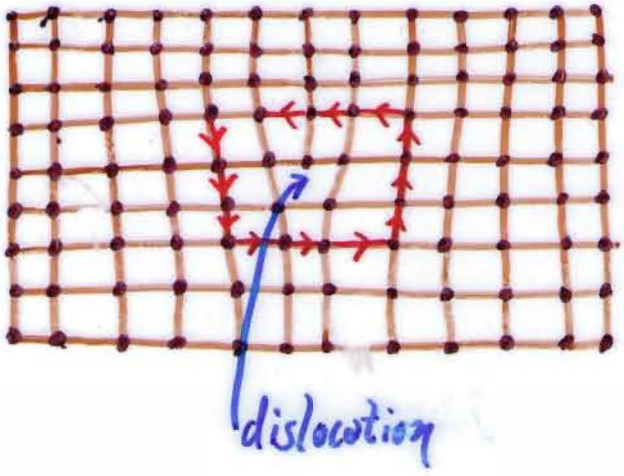
energetically desirable local structure:

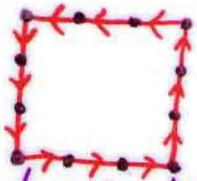


what about a crystal which grows like ?



we ultimately find spots like:

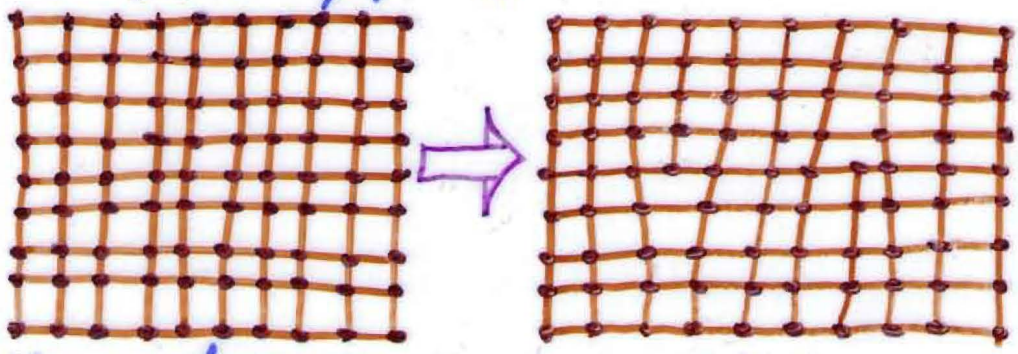




path closes in perfect crystal amount by which it does not in encircling dislocation is the Burgers vector ←

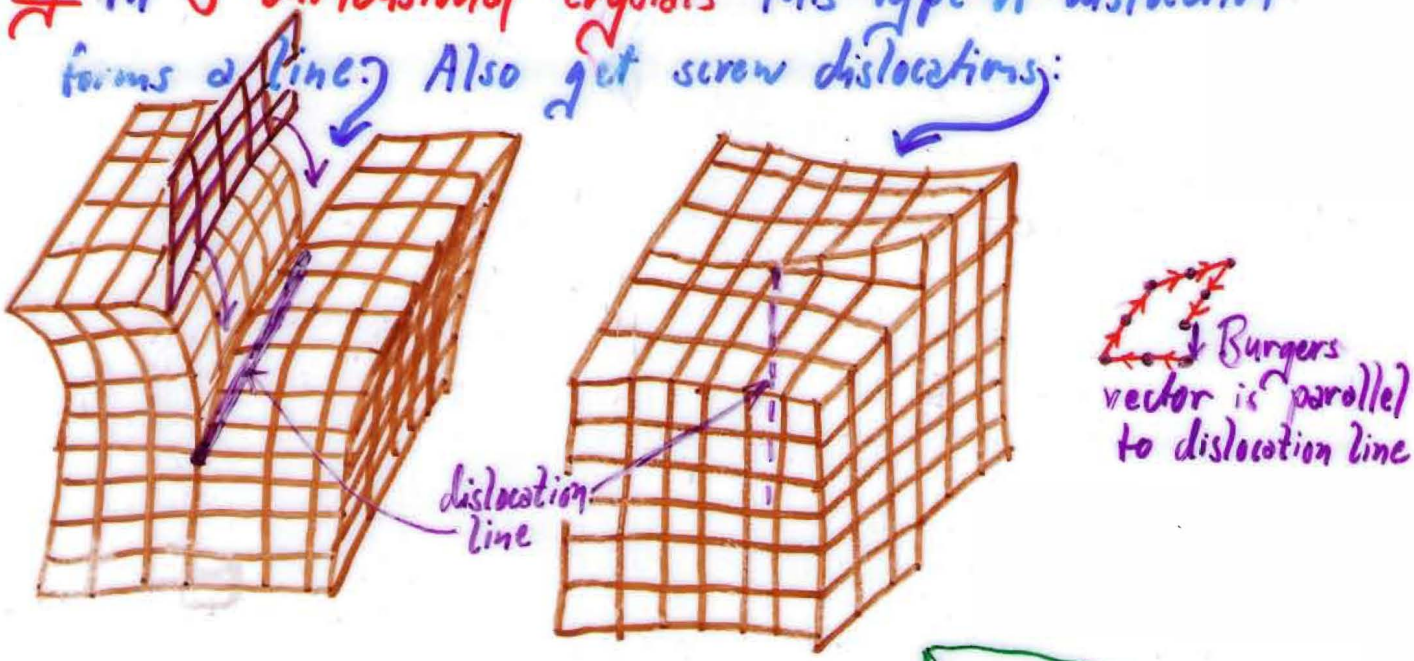
Response to external shear by moving dislocation:

From a perfect crystal, one can create a pair of dislocations with oppositely pointing Burgers vectors by an internal shear:



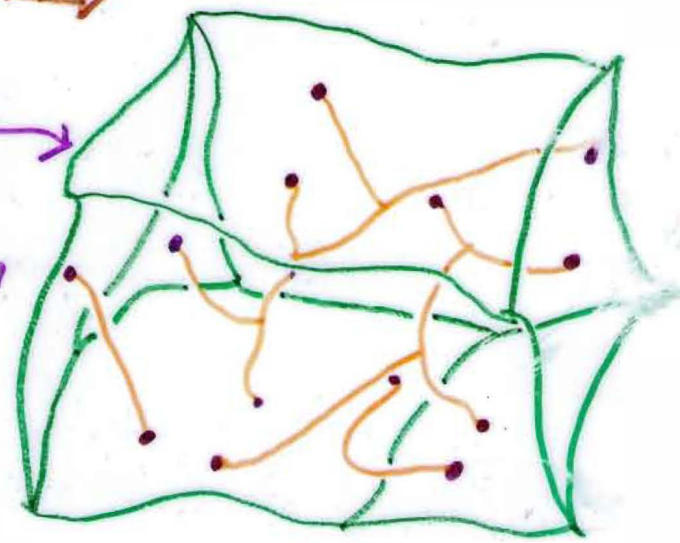
Crystals respond to stress by moving dislocations.

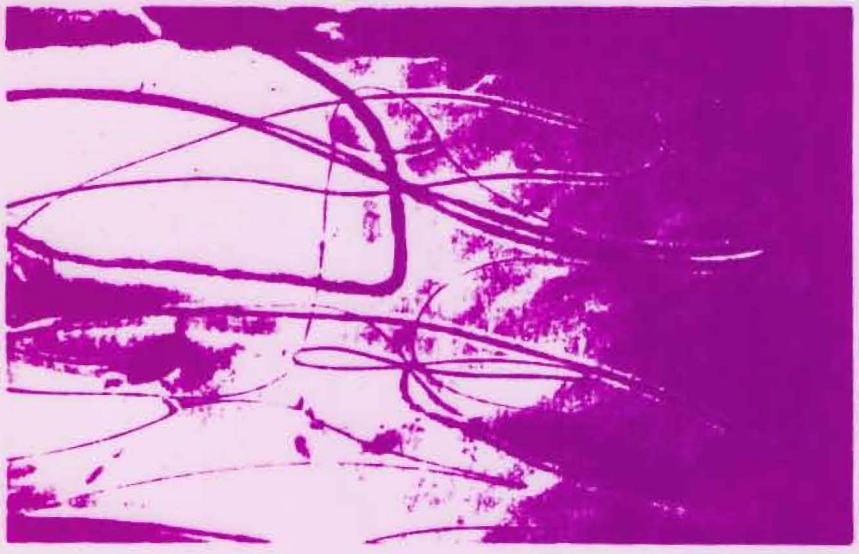
Eg In 3-dimensional crystals this type of dislocation forms a line. Also get screw dislocations:



Real crystals are made of grains spanned by a web of dislocations.

Structural properties result from motion of dislocations.





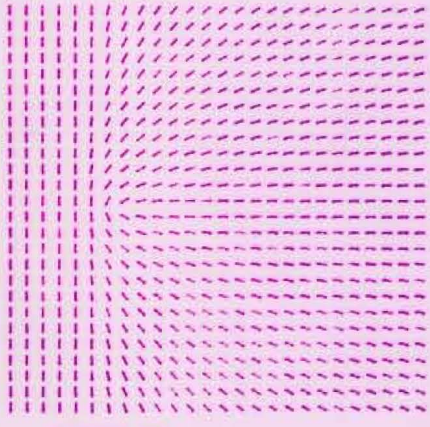


FIGURE 7B

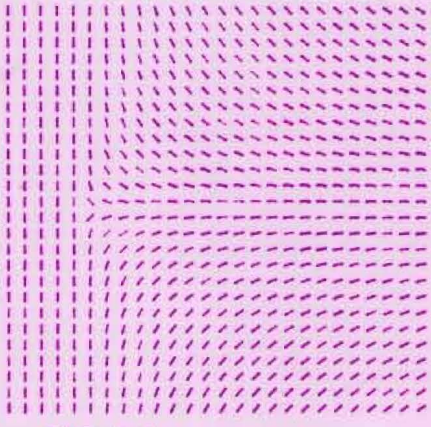


FIGURE 7C

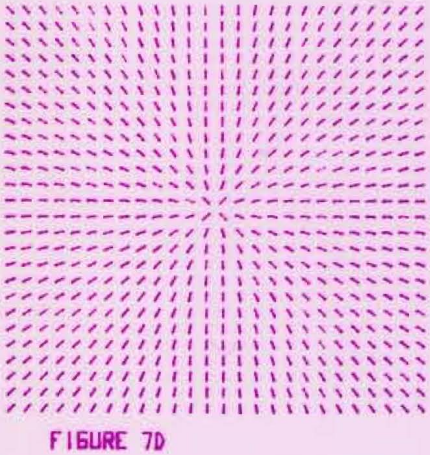


FIGURE 7D

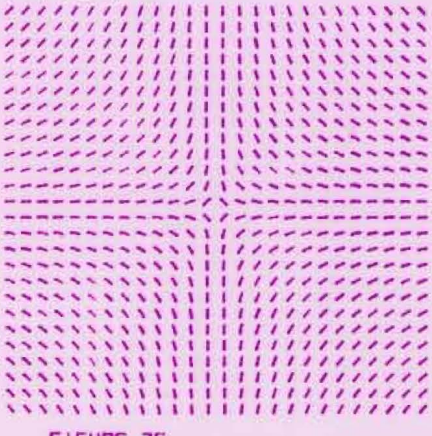


FIGURE 7E

FIG. 7. Line defects in nematics. This photograph has been made by M. Kléman. Here we see clearly two kinds of line defects: the thin lines, which represent topologically stable defects (index $\pm 1/2$), and the thick lines with bright sides, which represent topologically unstable defects with a probable index ± 1. We have defined in Ftn. 26 the Brouwer degree of a map $S_n \to S_n$: it is very intuitive in the simplest case $S_1 \xrightarrow{h} S_1$, since degree (h) is the algebraic number of turns on the circle S_1 of the image $h(\varphi)$ when the azimuth φ makes one positive turn (from 0 to 2π). Similarly one defines the index of the zero $v(x_0)$ of a continuous vector field on a two-dimensional surface: in any local coordinate system in the neighborhood of x_0, it is the degree of the correspondence h: azimuth of x (near x_0) $\to$ azimuth of $v(x)$; these two azimuths are defined modulo 2π. In the case of a field of directors $n(x)$, corresponding to the local orientation (in a plane) of a nematic, the azimuth of $n(x)$ is defined modulo π and the custom in physics is to define the index of a zero $n(x_0)$ multiples of $1/2$. Examples of index $1/2$, $-1/2$, 1, and -1 are given in Fig. 7(b), 7(c), 7(d), and 7(e), respectively. These figures represent the field of orientation of a nematic near a line defect and in a plane perpendicular to it. The index is conserved when the perpendicular plane is moved along the line defect. It is a good

$\pi_0([G:H]) = 1$: there are no wall defects. This is the case for superfluid states and more generally for all liquid or liquid crystal phases; as a general rule singularity surfaces in liquid do not seem to be stable, as is predicted by the homotopy classification for mesomorphic phases made with chiral molecules. In these cases the isotropic liquid phase has a rotating power for optical polarization, so its symmetry group cannot contain reflections: it is $E_0(3)$, the connected Euclidian group, generated by rotations and translations. These optically active chemical compounds often have chiral mesomorphic phases (chiral smectic C or cholesterics), whose rotating power can be larger by several orders of magnitude than that of the isotropic phases. Homotopic classification would predict wall defects if nonchiral molecules [$G = E(3)$, e.g., racemics] could form chiral phases [i.e., with a global symmetry group $H \subset E_0(3)$]. No such phases have yet been observed: they seem to be fundamentally unstable. The situation is very different for crystals. Crystal atoms or ions have no chirality. Among the 230 crystallographic classes in three dimensions, 65 correspond to crystallographic groups $H \subset E_0(3)$ [among these 65 there are 11 pairs of isomorphic groups, conjugate in Aff(3) but not in $\mathrm{Aff}_0(3)$]. Crystals belonging to these 65 crystallographic classes do have topologically stable wall defects, called "twin boundaries": indeed they annihilate by pairs as required by $\pi_0([E(3):H]) = Z_2$.

We define at the end of Sec. II. A "topologically stable configurations." The continuous function $\phi: V \to [G:H]$, describing the local states of the medium in the volume V, is defined everywhere on V but it is not homotopic to a constant. This can happen when the domain V occupied by the phase is not contractible—for instance, if V is the volume between two coaxial vertical cylinders, bounded by two horizontal planes. Consider this volume filled with a nematic whose molecular long axis (the director) is horizontal and everywhere orthogonal to the horizontal radial direction: hence on the boundary ∂V this axis is tangential; this is actually the case in many physical situations where the nematic molecules prefer to be parallel to the wall of the container. We have described a configuration; it is topologically stable and has no defects. If V is contractible, we have proved, in the same fashion as Eq. (29), that every continuous function $V \xrightarrow{\phi} X$, valued in an arbitrary topological space X, is homotopic to a constant. This would exclude the existence of topologically stable configurations if we were not *imposing some boundary conditions* (making V equivalent to a noncontractible space) which are naturally suggested by physics. Here we define and discuss what we shall call the "normal" configurations: those which may exist inside an ordered medium otherwise in a perfect state. Consider a (contractible) domain V occupied by an ordered phase in a perfect state everywhere outside a ball B_3 of radius a, so that the function $\phi: V \to [G:H]$ describing the local states is constant on $V - B$. Of course V can be extended (continuously) over B with the same constant value. Given a function ϕ, constant on the boundary $\partial B_3 = S_2$ of the ball B_3, are there any ways to extend it continuously inside B_3 which are not homotopic to the constant? Since the ball B_3, with all points of its surface identified, is homeomorphic to the sphere S_3, the homotopy classes of the continuous extensions of ϕ are given by $\pi_3([G:H])$. For mesomorphic phases all the $\pi_3([G:H])$ are isomorphic to Z (see Table II).

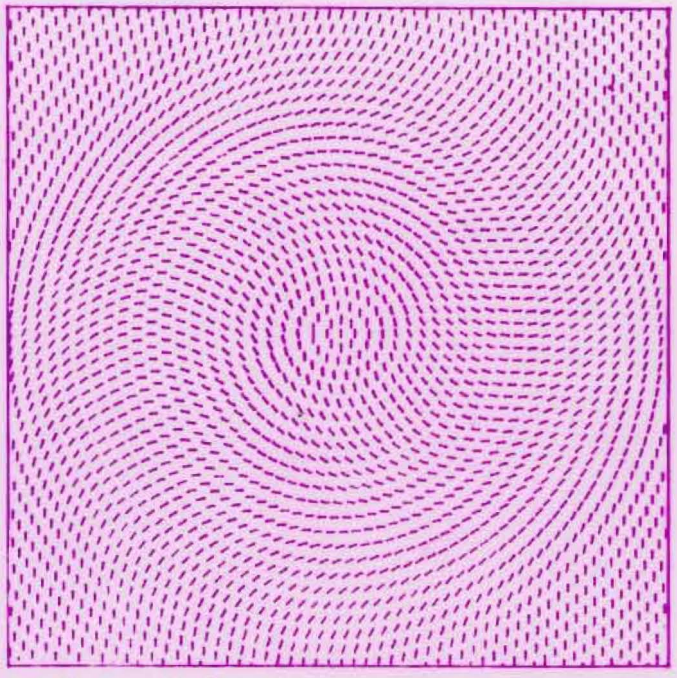


FIG. 8. Example of a configuration in a two-dimensional "nematic" vertically oriented outside the configuration. With the origin of coordinates at the center of the figure, the orientation at the point $\mathbf{r}$ makes an angle $f(r)$ with the vertical, where $f(0) = 0$ and $f(r) = \pi$ for $r \geq a$. This configuration is not topologically stable; it can be continuously transformed back into the perfect "nematic" vertically oriented. Note indeed that $[G:H] = S_1$ and $\pi_2(S_1) = 1$.

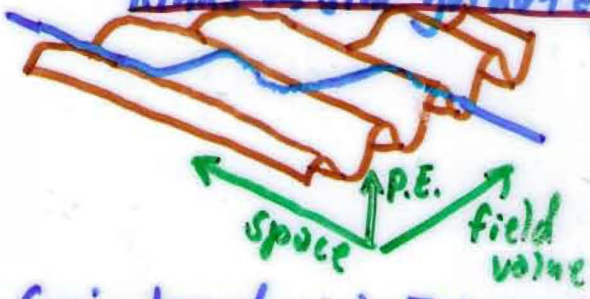
Here is a classical construction of a configuration belonging to the homotopy class $n \in Z$. Take as the origin of coordinates the center of B_3, the ball of radius a. Choose a continuous function $f(t)$ such that $f(0) = 0$ and $f(t) = 1$ for $t \geq 1$; such a function can even be infinitely differentiable, e.g.,

$$f(t) = \begin{cases} 1 - e^{-t^2/(1-t^2)} & 0 \leq t \leq 1 \\ 1 & 1 \leq t \end{cases} . \tag{26}$$

Then, the position of the local state at the point $\mathbf{r}$ is obtained from that of the local state everywhere constant in $V - B_3$ by a rotation of axis $\mathbf{r}$ and angle $2\pi n f(r/a)$.[37]

As Table II shows, the values of π_3 homotopy groups predict no such configurations in superfluid ^{4}He but do predict them for the two superfluid phases of ^{3}He. These configurations may have been observed, but not the line defects, yet.

[37]We have just defined smooth maps $S_3 \xrightarrow{g_n} P(R,3) \sim SO(3)$. The reader can easily verify, from the definition we gave of the Brouwer degree in Ftn. 26, that $\deg(g_n) = 2n$. Since S_3 is the universal, double covering of $P(R,3)$, the homotopy class is n. We could have defined $g_n = \theta \circ h_n$: $S_3 \xrightarrow{h_n} SU(2) \xrightarrow{\theta} SO(3)$; in Appendix G we define θ and we give an explicit identification of S_3 (given by B_3 with all boundary points identified) with the elements of SU(2). With this identification, h_1 is the identity map and $\deg(h_n) = n$.

Space	Homotopy Group	Dislocation	Eg
1-dim	π_0	point	kinks in sine-gordon eqn. 
2-dim	π_0	line	Grain boundary in Ising spins 
	π_1	point	 vortices, flux lines Rayleigh-Bénard defects
	π_2	non-removable configuration no defect	smectic
3-dim	π_0	planar	crystal grain boundaries
	π_1	line	crystal dislocations etc.
	π_2	point	monopole, hedgehog in field theories
	π_3	configuration	Finkelstein's magnetic kink in MHD
4-dim	⋮	similar to above	
	π_3	point	instantons

Statistical Properties can be completely dominated by defects – they are deep local minima in energy.

Might be dangerous in stochastic search algorithms: eg. in travelling salesman

true soln

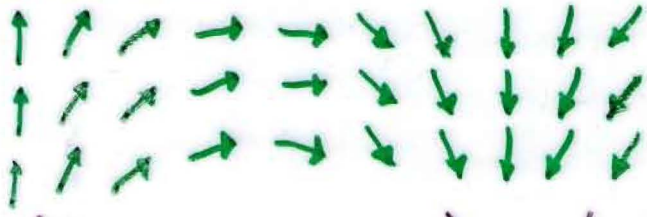
topological barrier to true soln.

Phenomena like in spin glasses with frustration.

Eg. 2-dimensional Heisenberg spin system

Model for ferromagnet $\quad E = -J\sum_{\langle ij\rangle}\cos(\phi_i - \phi_j)$

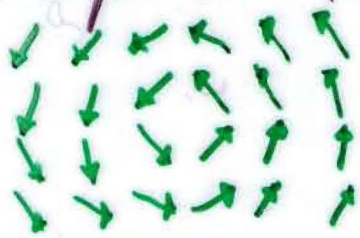
Spin waves:



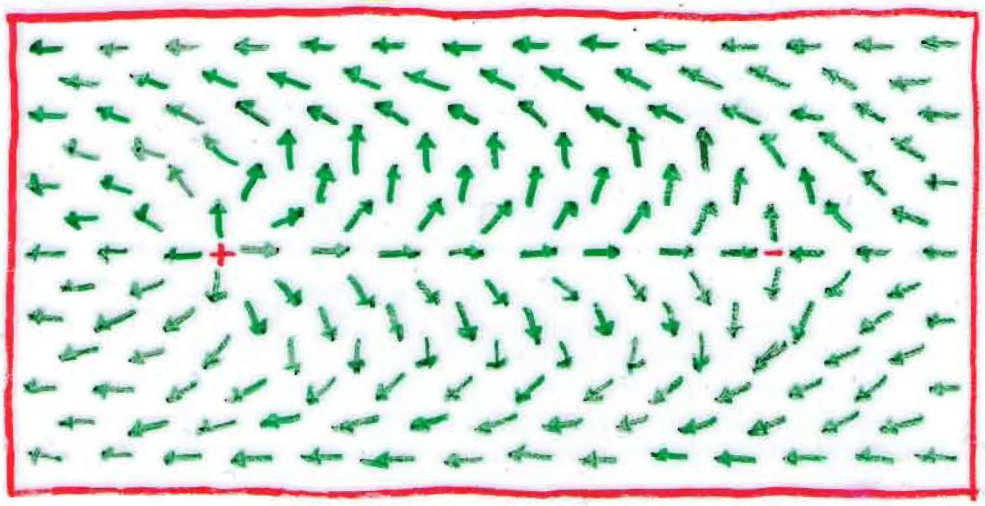
$\Rightarrow$ No spontaneous magnetization

Yet there is a phase transition!

Vortices:



Vortex-Anti-Vortex Pairs:



Kosterlitz + Thouless Picture

Vortices have same statistical mechanics as a coulomb gas of charged particles.

Model for Solid $\rightarrow$ Liquid transition!

Winfree – "The Geometry of Biological Time"

Phase singularities in: color vision, time on earth, tides, fruitfly circadian rythm, morphogenesis, cuticle, muscle fiber, slime molds, waves in brain, eye and:

SUDDEN DEATH HEART ATTACKS:

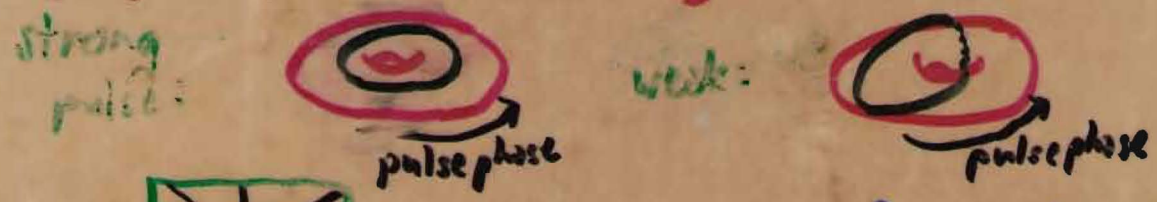
Heart cells have phase – i.e. state space is a circle

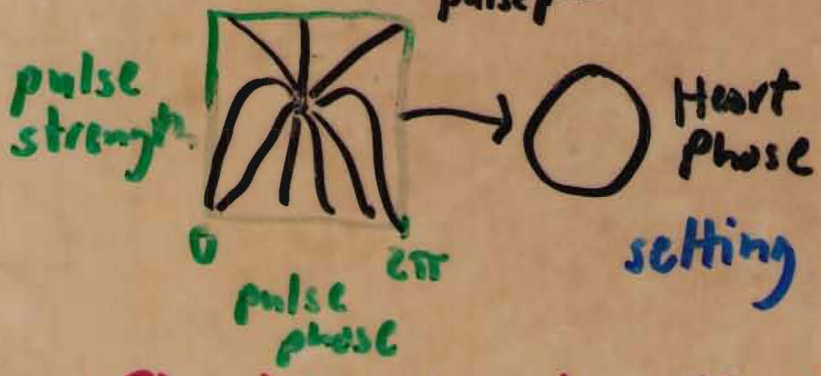
fibrillation = phase singularity = point on surface of heart where phase winds around 2π in encircling it:

⇒ spiral waves ⇒ BAD for owner (Mines 1914)

weak pulse shifts phase slightly, strong pulse resets the heart

Graph pulse phase vs. resulting heart phase on torus





Must be pulse height and phase which is arbitrarily close to setting heart cells to anything.

Shocking heart with such a pulse is likely to set up spiral waves causing spiral waves of death.

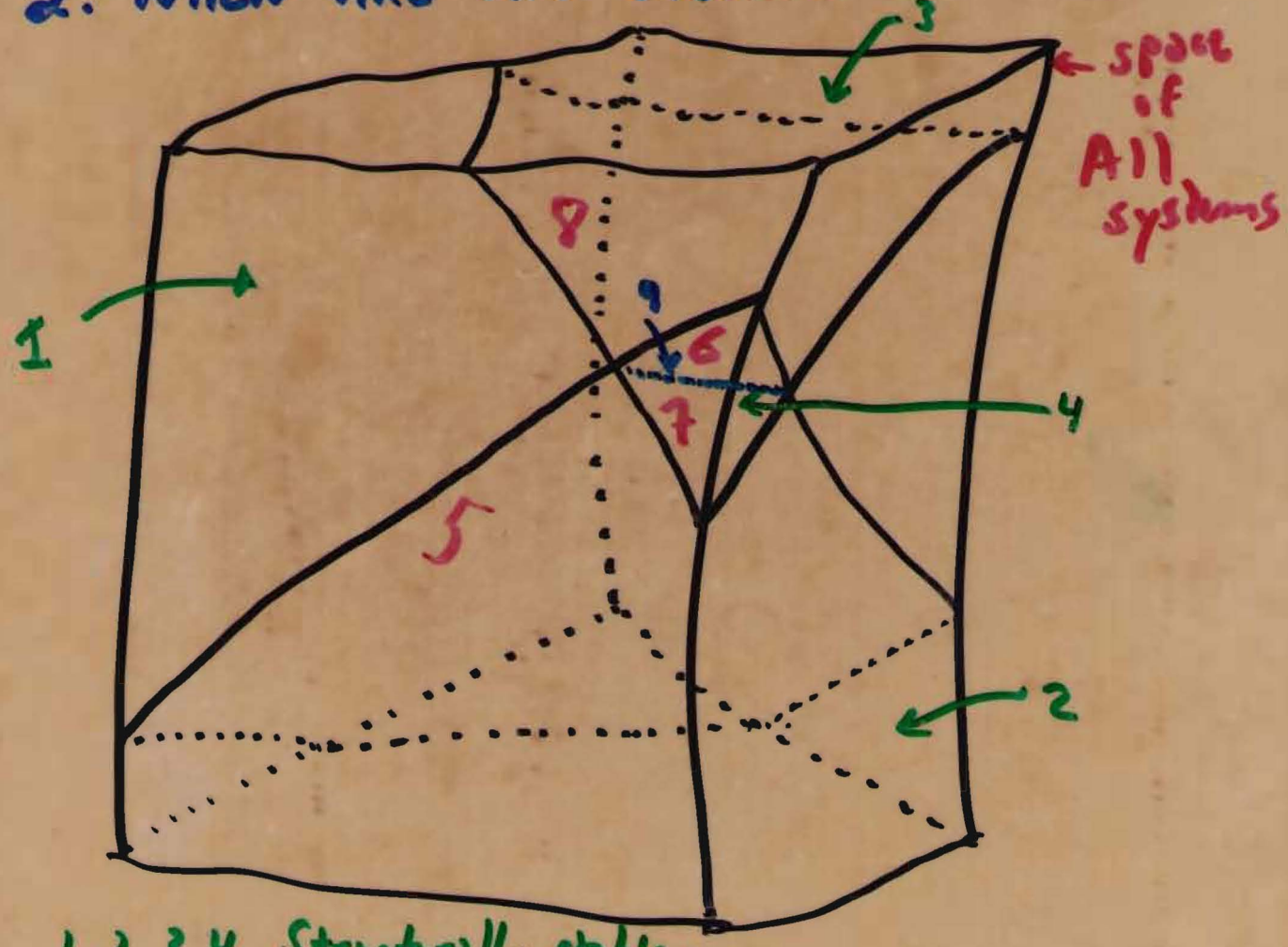
ANDRONOV AND PONTRJAGIN (1937):

STRUCTURAL STABILITY

The only physically relevant aspects of a model are those which are preserved under small perturbations.

Need to specify:

1. THE ALLOWED PERTURBATIONS
2. WHEN ARE TWO SYSTEMS EQUIVALENT?



1, 2, 3, 4 Structurally stable

5, 6, 7, 8 Codimension 1

9 Codimension 2

Related notion due to Smale (~1962):

Genericity

A property of a class of systems is generic if it is true for almost all of them.

(Almost all often means: a countable intersection of open dense sets)

In some situations the property of being structurally stable is generic.

(but not for dynamical systems on state spaces of dim 4 or greater)

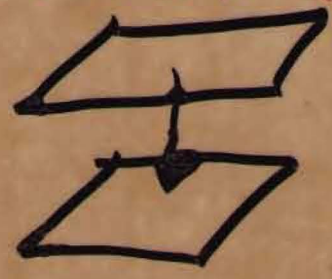
EXAMPLE (Relevant to vision):

In 1955 Whitney showed there were only 3 structurally stable classes of maps from the plane to the plane. (locally)

Perturbations: to nearby smooth mappings

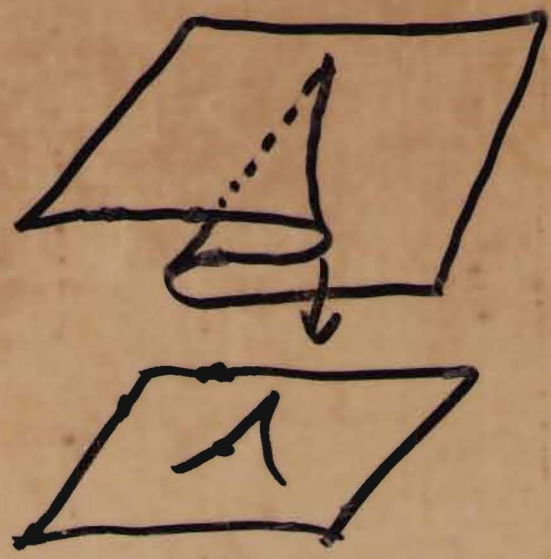
Equivalence: 2 maps equiv. if can smoothly change coordinates to turn one into the other.

the identity:

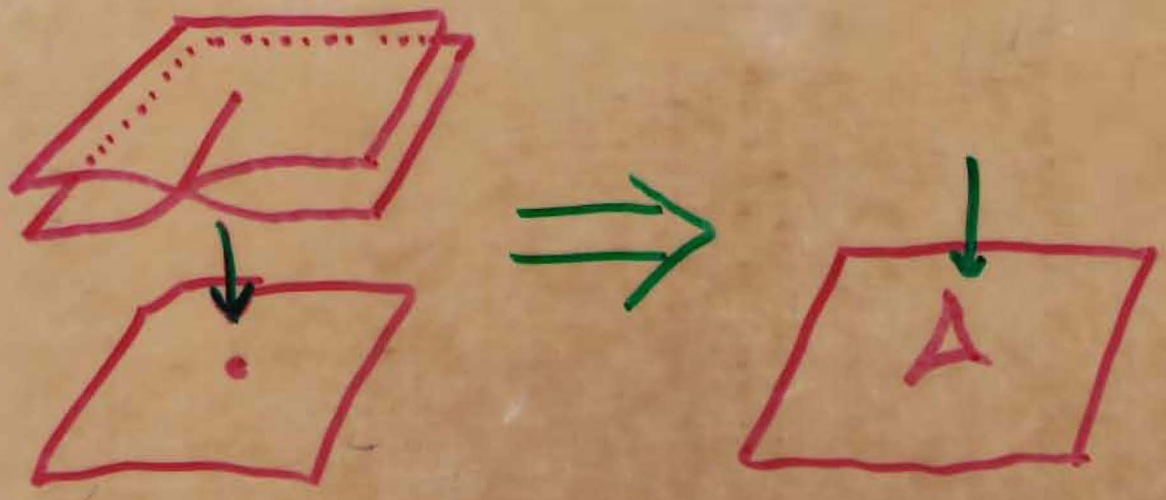


the fold:

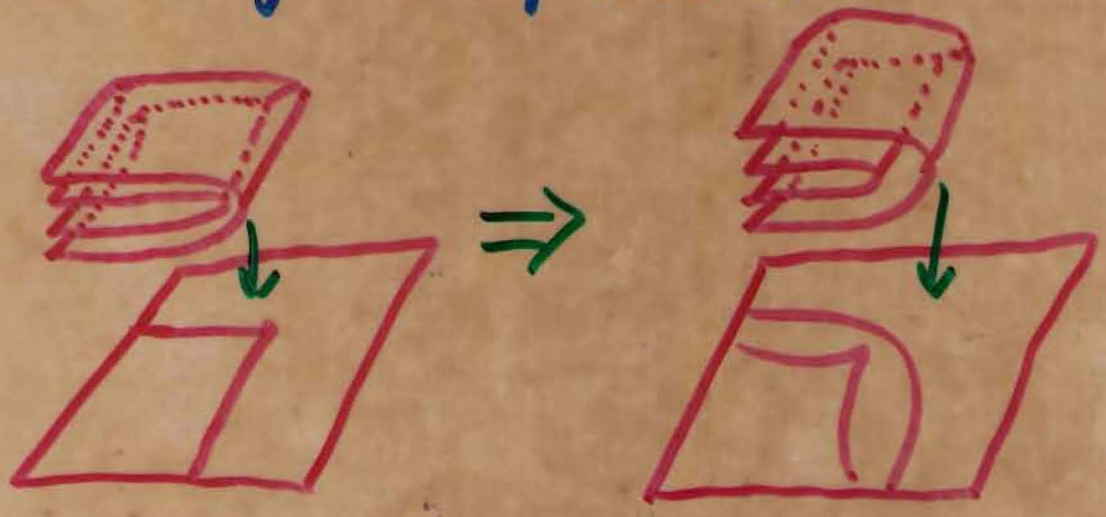
the cusp:



Example: Perturb the Riemann surface for $\sqrt{\ }$ slightly to get 3-cusps:



Example: Perturb the "folded handkerchief" map to get a cusp and a fold:

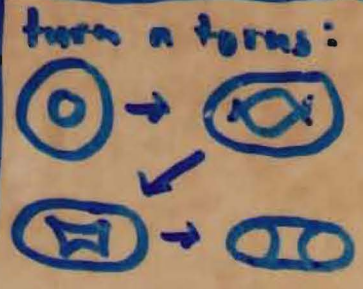


Look at a rumpled shirt for other examples.

Clear torus:

Remove hidden line:

turn a torus:

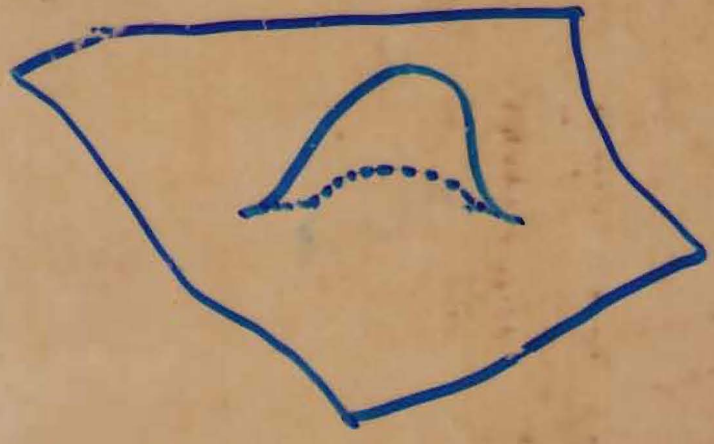


KOENDERINK AND VAN DOORN 1982

simple model of vision: maps from smooth (2-d) surfaces of objects to 2-d surface of retina.

singularities of mapping = places where line of sight is tangent to object i.e. "edges" of object.

fly over a smooth hillside ⇒ get "lips" event:



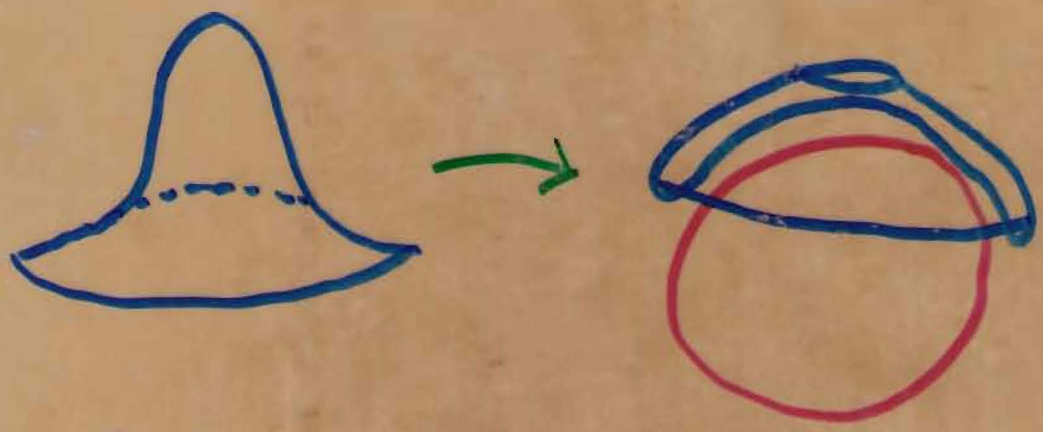
edge of bulge ends at two cusps → must bend outward

A study of artists' depictions of breasts in nudes from the 15th century to the present shows that the edges are always drawn bending inward.

Figure 6. Picasso: *Mr crouchd* [illegible].

Artist [illegible] the parabolic line on the surface.

Consider the Gauss map from a surface to the unit sphere, sending a point to the unit normal there:



Positive curvature - mapped preserving orientation
Negative curvature - mapped reversing orientation
Parabolic curves - the folds of the Gauss map

For homogeneous surfaces, the observed intensity is a function only of surface orientation.

⇒ Intensity is a smooth function on sphere.

→ Parabolic lines will generically be maxima or minima and so stand out.

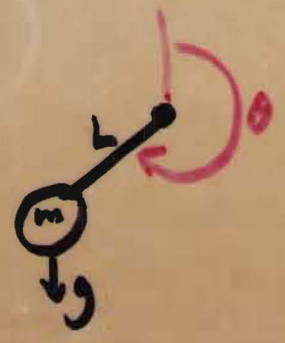
Moral: Things are only structurally stable in the right class of systems.

DYNAMICS

Example: Damped pendulum

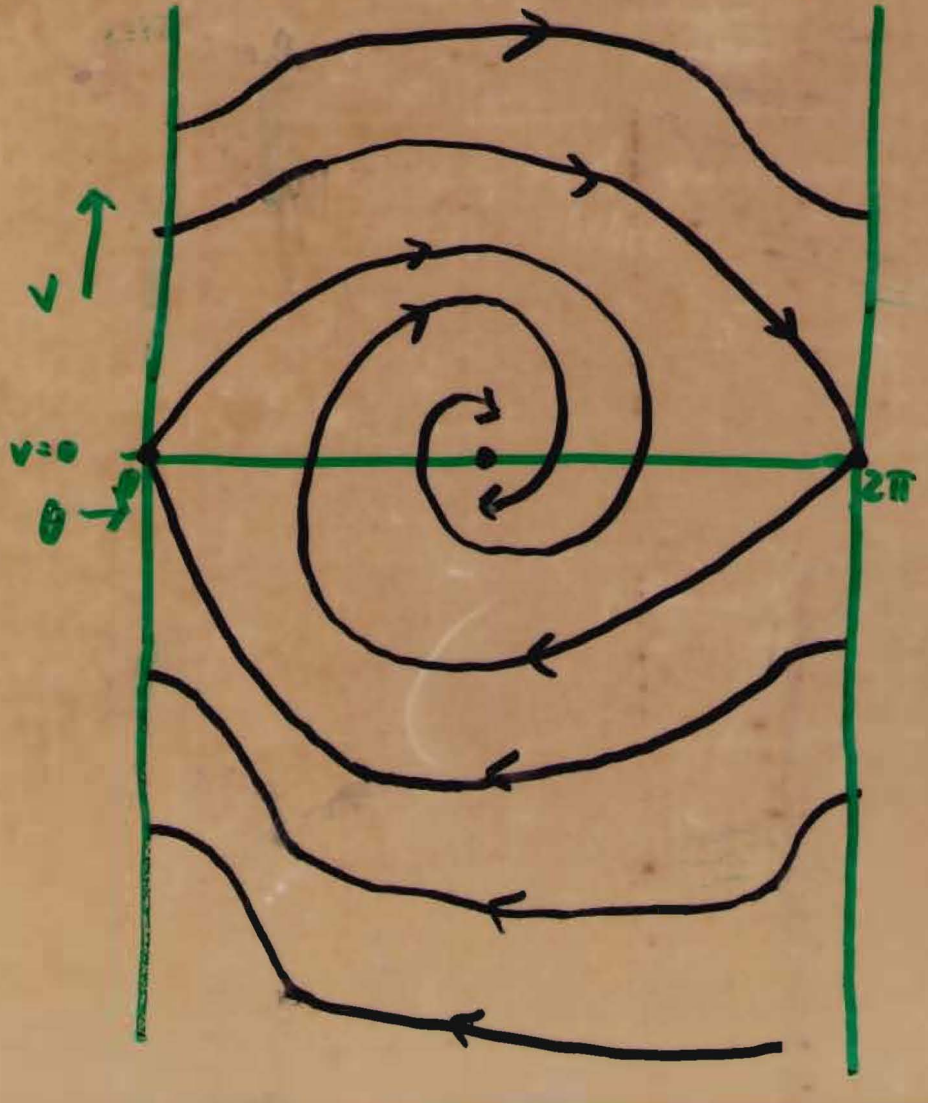
$$\ddot{\theta} = \frac{g}{L}\sin\theta - \alpha\dot{\theta}$$

or $\begin{cases} \dot{\theta} = v \\ \dot{v} = \frac{g}{L}\sin\theta - \alpha v \end{cases}$



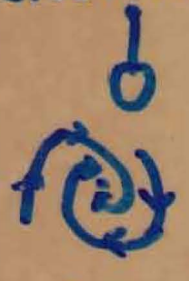
State space is a cylinder

Open up the cylinder to see the dynamics:



Two fixed points:

stable sink



unstable saddle

structural stability of the dynamical vector field:

equivalence relation: 2 v.f.'s are equivalent if $\exists$ continuous change of coordinates taking orbits of one to orbits of the other

allowed perturbations: can add any smooth vector field.

Flow box theorem (straightening out theorem):

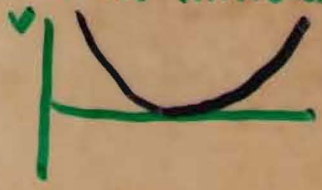
In a neighborhood of a point where a vector field doesn't vanish, we can straighten it out by a smooth change of coordinates.
(so, away from fixed points, all v.f.'s are locally equivalent)

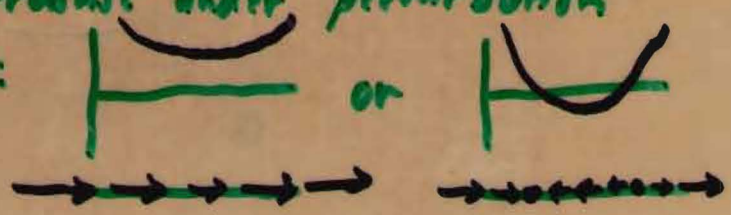
In 1-dim:  and 
are structurally stable.

 is not.

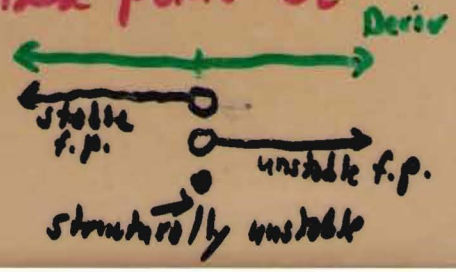
Can see by graphing:

these intersect 0 transversally → robust under perturbation



perturbs to: 

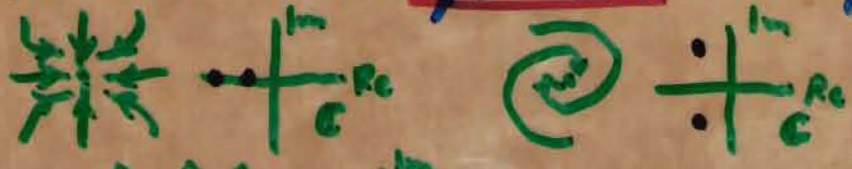
Condition for structural stability is that the derivative of the vector field at the fixed point be positive or negative, but not zero.



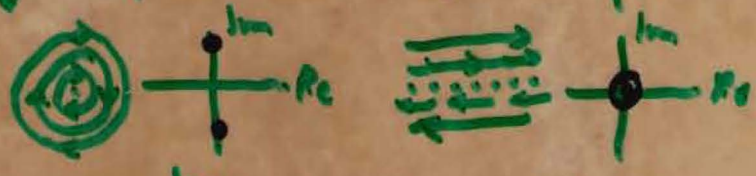
For higher dimensions, the eigenvalues of the linearization of the v.f. at the fixed point determine whether it is structurally stable.

Hartman's theorem: If no eigenvalues are on the imaginary axis then structurally stable and equivalent to its linearization. (denoted a hyperbolic fixed point).

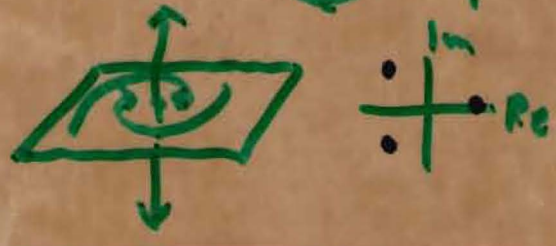
2-d examples:



Not structurally stable:



3-d example:



# of e-values on left half plane = dimension of points asymptotically attracted to f.p.

this set is called the stable manifold

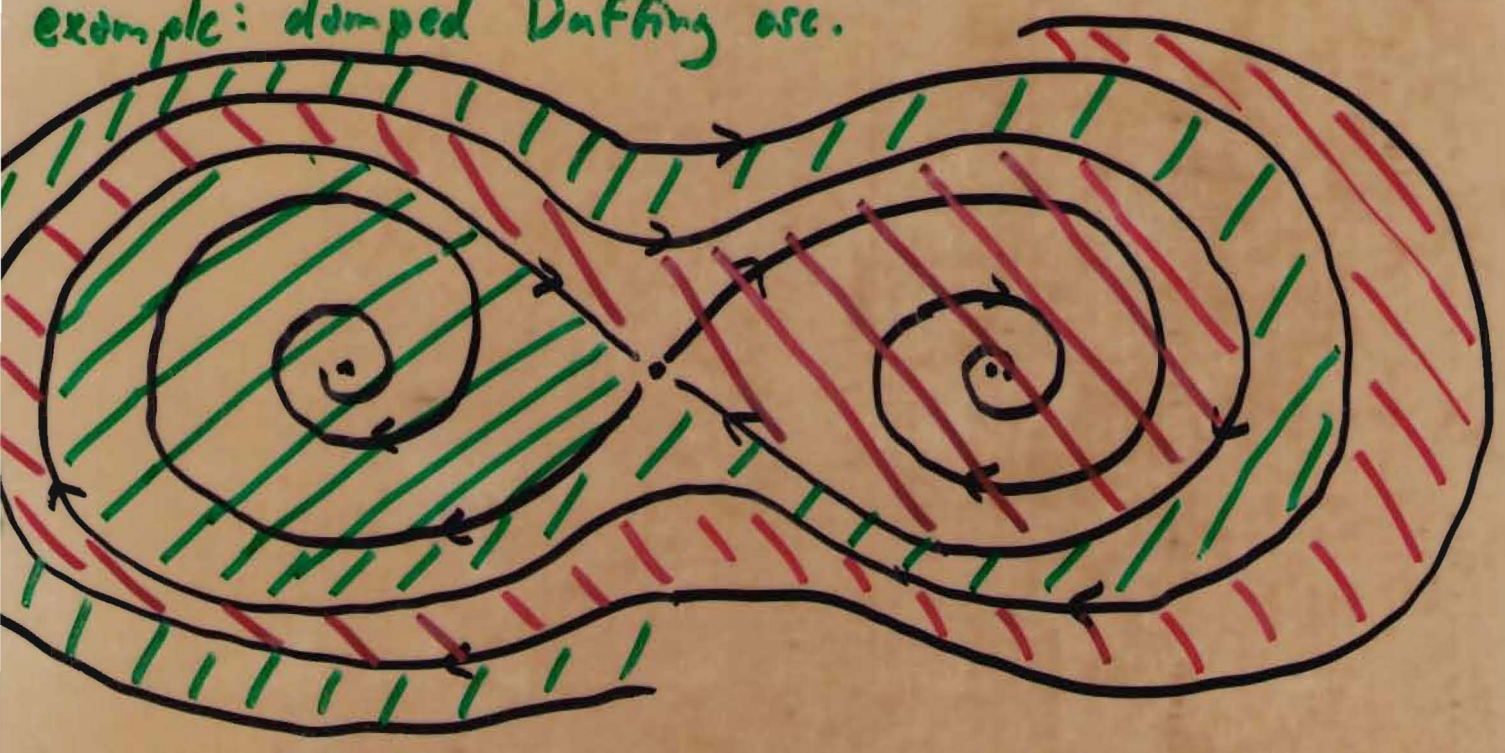
# of e-values on right half-plane = dimension of set of points asymptotically attracted as time goes backward.

this set is called the unstable manifold.

Stable + Unstable Manifold Thm: These sets really are manifolds, are immersed copies of Euclidean space, and are tangent to the corresponding eigenspaces.

For an attracting fixed point (all e-values on the left) the stable manifold is called the basin of attraction.

example: damped Duffing osc.



The stable manifold of the saddle seperates the two basins of attraction and so is called a separatrix.

Periodic Orbits:

Poincaré section:

Poincaré return map:

periodic orbit is fixed point of return map.

look at eigenvalues of linearization around fixed point.

(map linearization) = $e^{\text{(vector linearization)}}$

$\Rightarrow$ role of imaginary axis in $\mathbb{C}$ for v.f. $\Leftrightarrow$ unit circle in $\mathbb{C}$ for maps.

inside unit circle = attracting directions

outside unit circle = repelling eigendirections

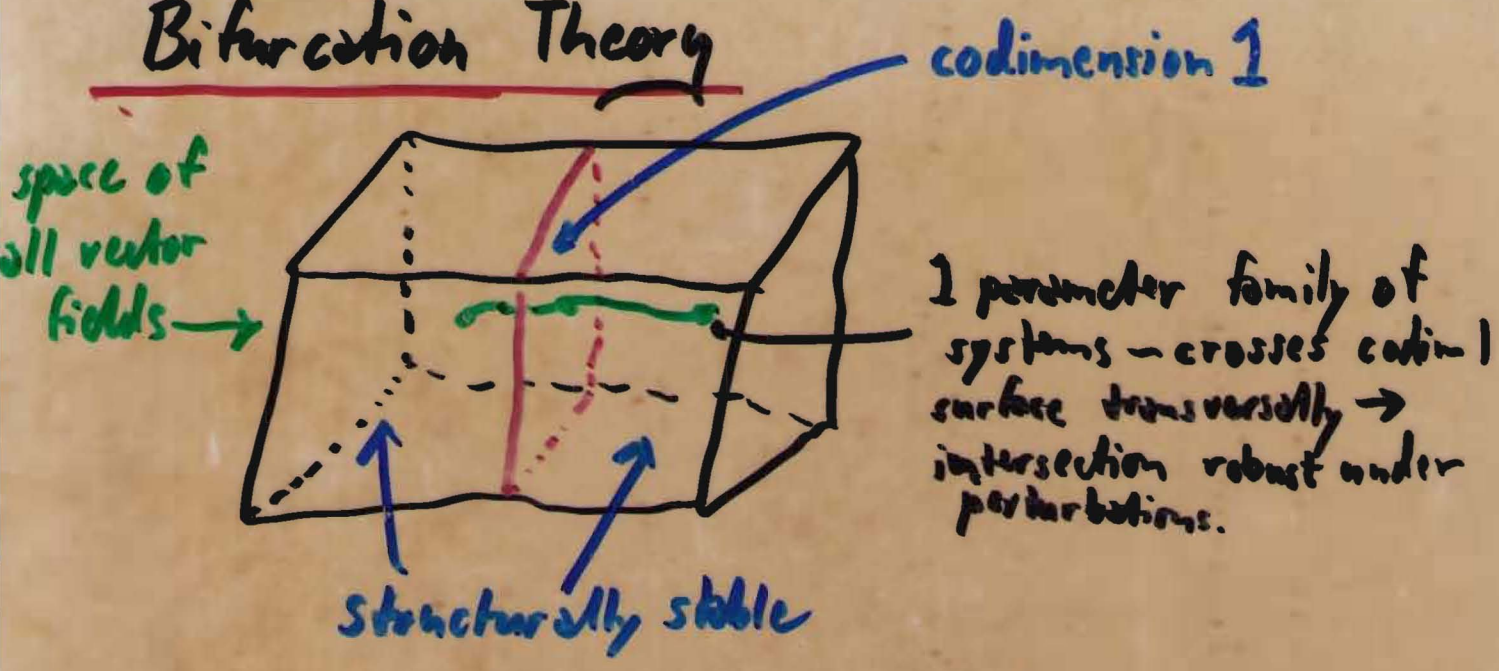
on the unit circle = structurally unstable.

again define stable + unstable manifolds:

If a periodic orbit is attracting it is called a limit cycle.

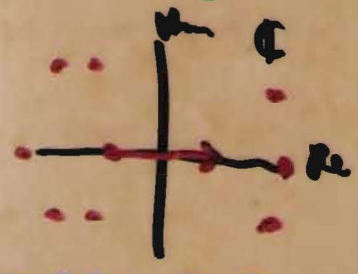
Its stable manifold is its basin of attraction.

Bifurcation Theory



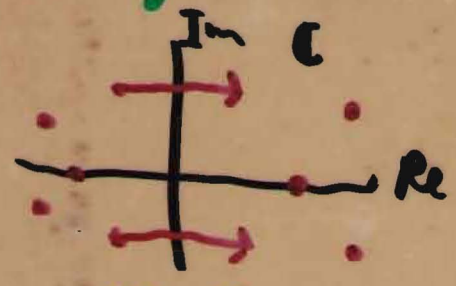
Eigenvalues move smoothly with parameters

⇒ fixed pt. is codim 1 if has 1 zero eigenvalue or 1 complex conj. pair

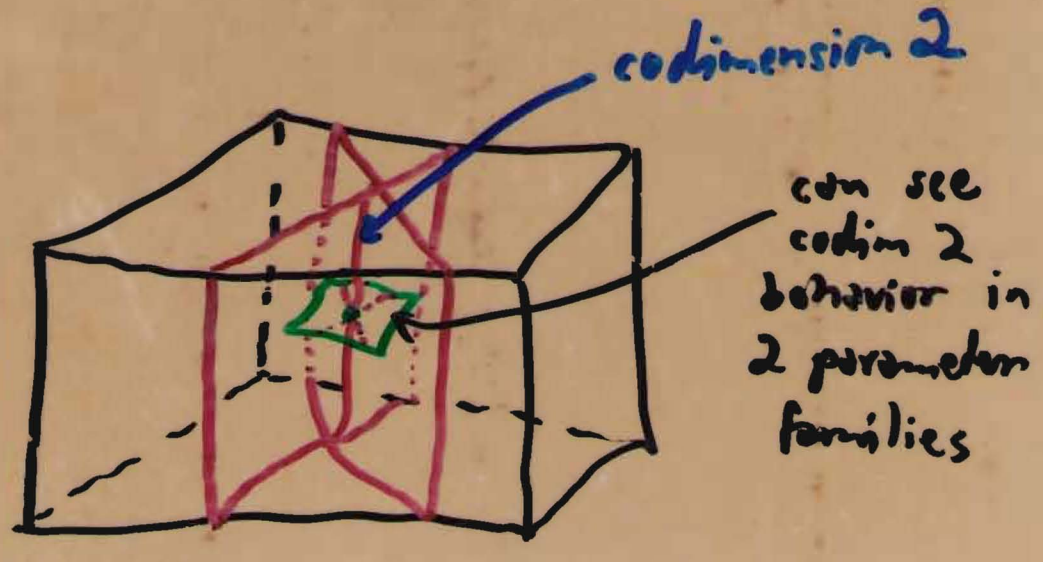


Saddle Node

or

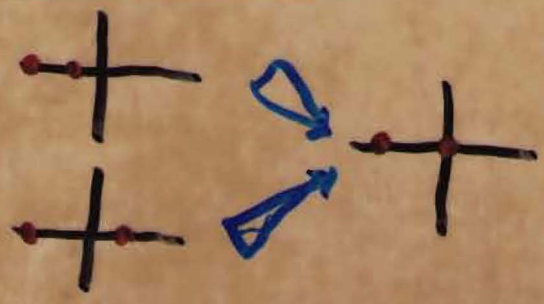


Hopf

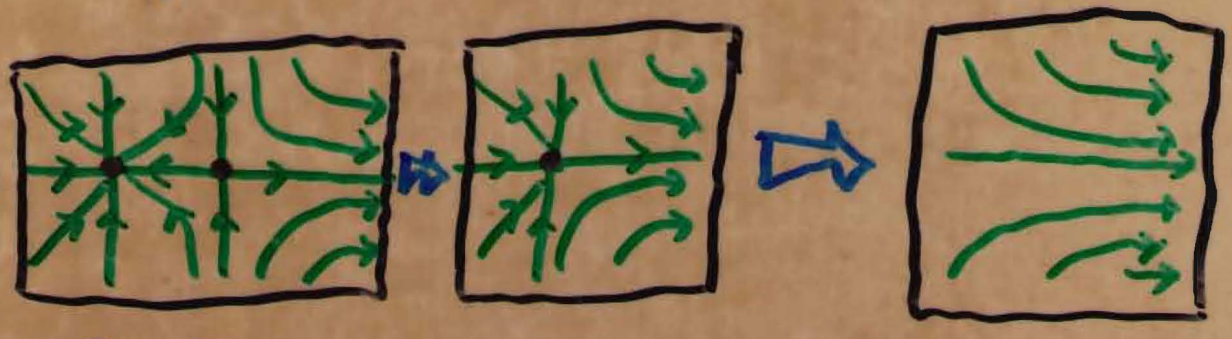


SADDLE-NODE BIFURCATION

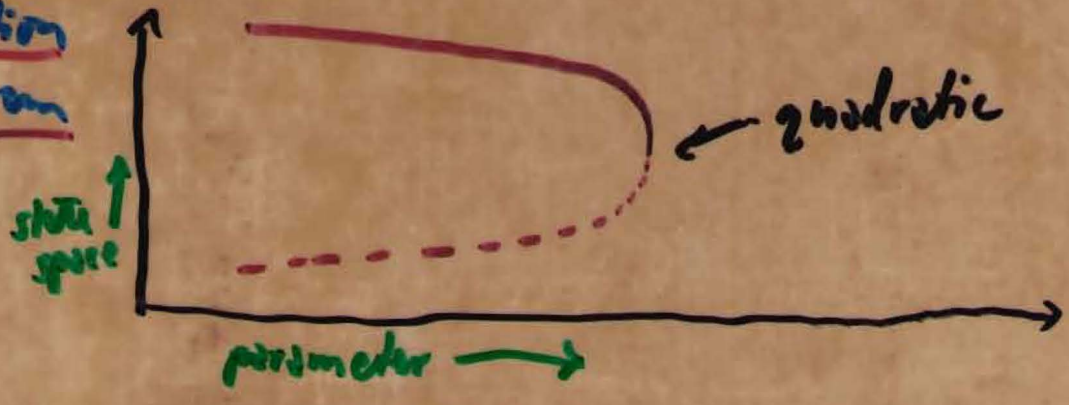
Eigenvalues



Phase portraits



Bifurcation Diagram



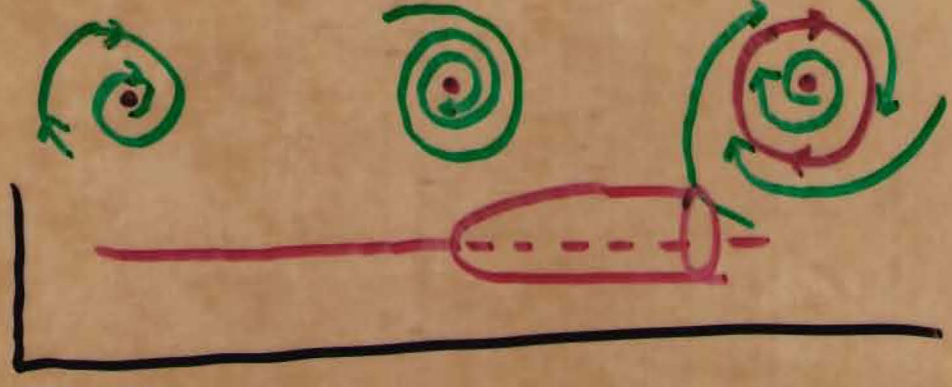
Example: intensity function of a scene – convolve with ever wider gaussians (run diffusion equation on image)

Gradient vector field bifurcates
hilltop & pass annihilate

HOPF BIFURCATION

Eigenvalues

Soft loss of stability



Hard loss of stability

Examples:

Waving Flag

River Meanders

Honey Motor Oil Water

CODIMENSION 2 BIFURCATIONS OF FIXED PTS.

There are five cases:

1. 2 zero evalues

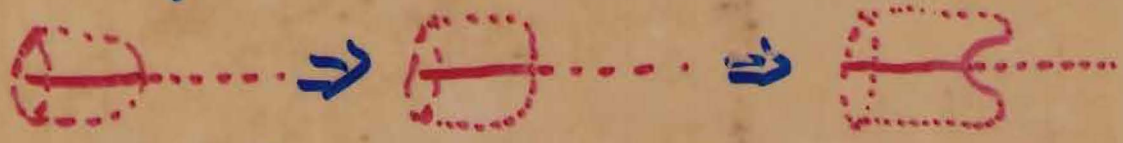
Bogdanoff

2. 1 zero & 1 c.c. pair

3. 2 c.c. pairs

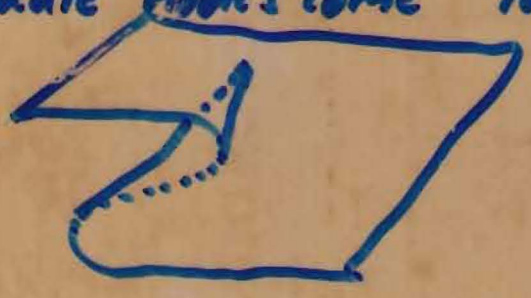
Not understood (interesting stuff occurs in 3 & 4 dimensions ⇒ extremely complicated)

4. Hopf changes from hard to soft:

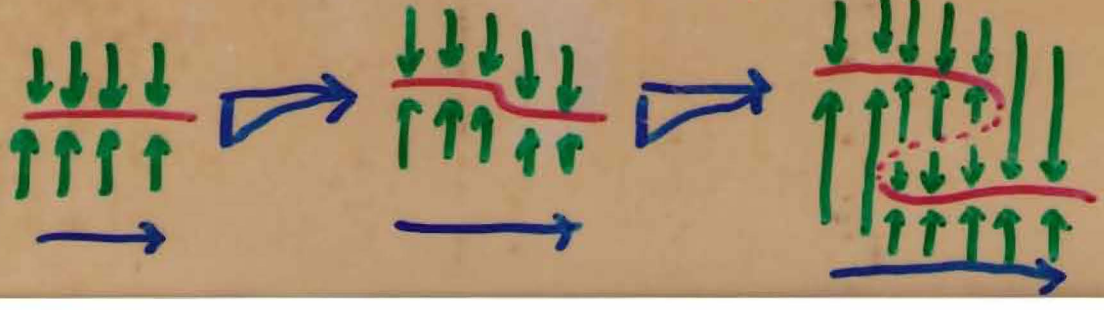


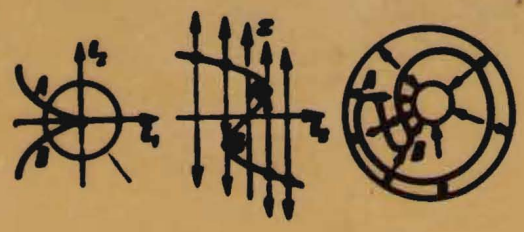
second parameter ⟶

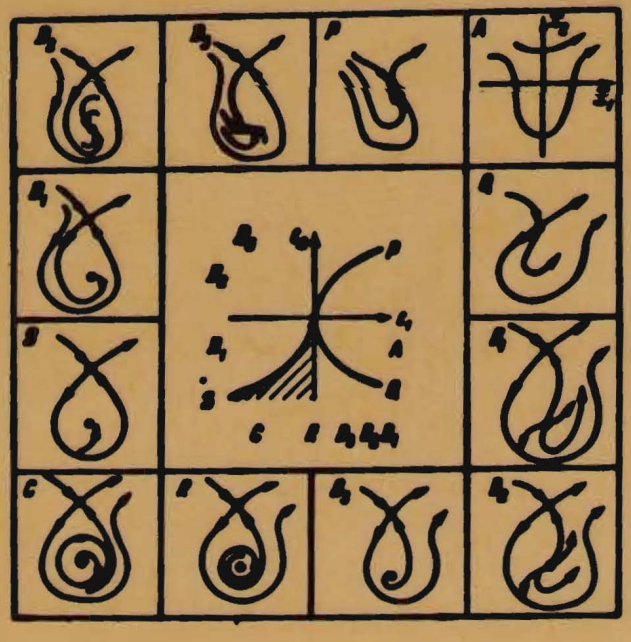
5. 2 saddle nodes come together



describes the creation of hysteresis





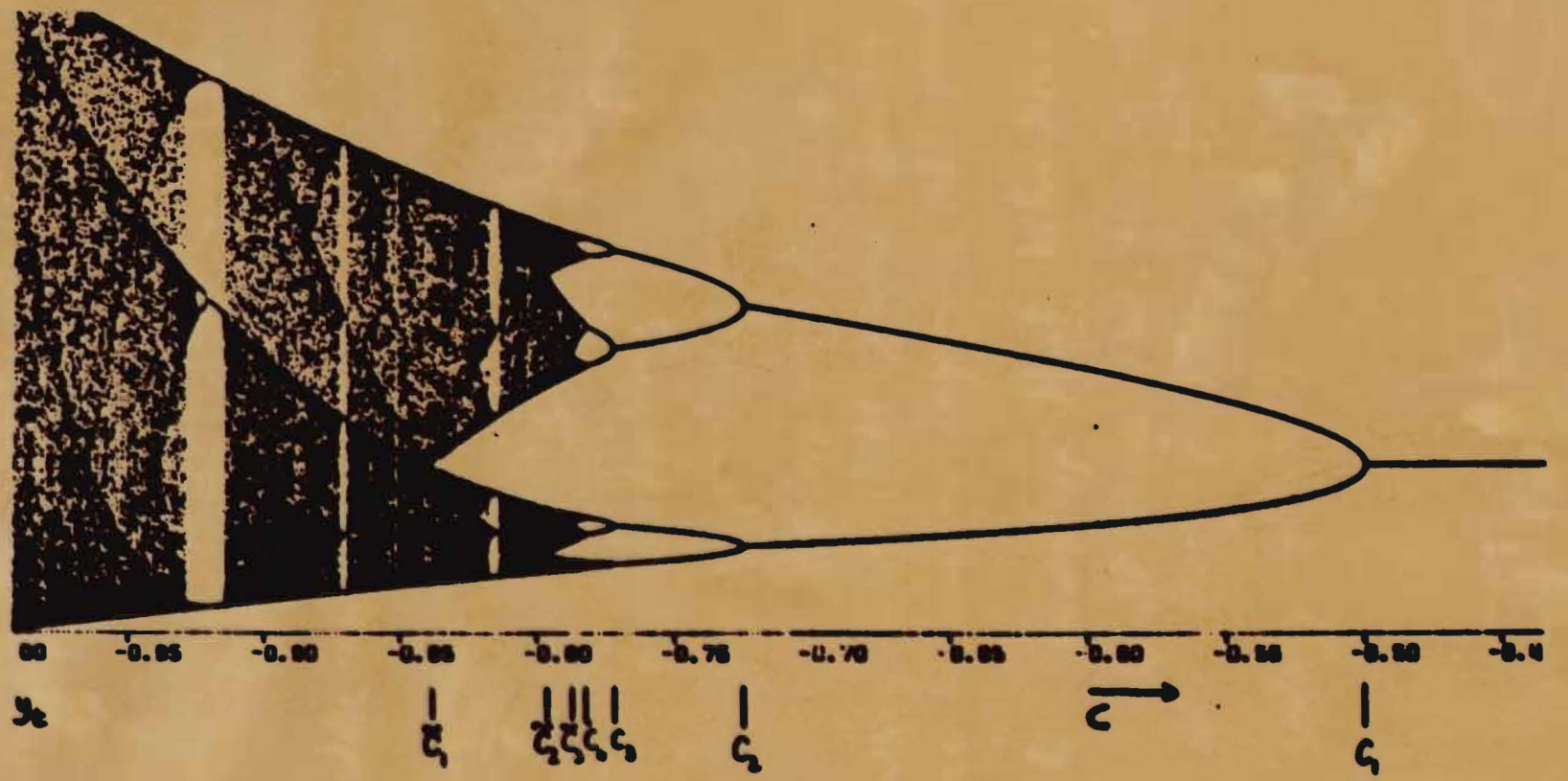


Quadratic Map

$x \to 2(cx + x^2)$

equivalent to logistic map

$y \to ay(1-y)$



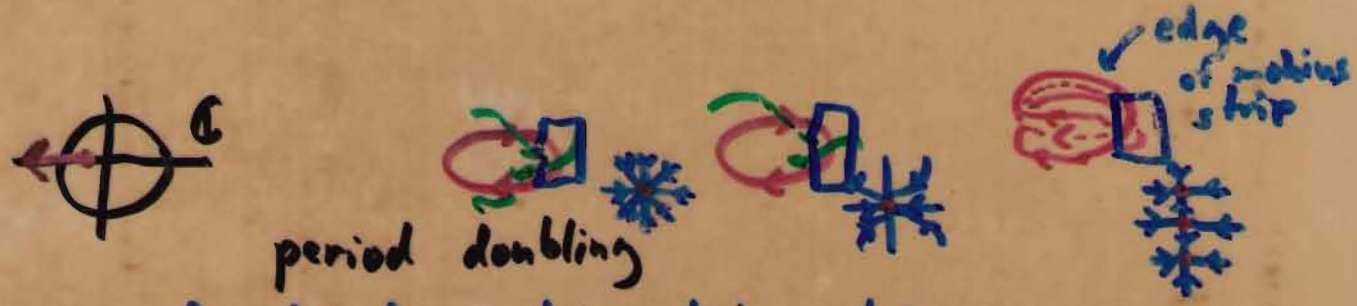
$$c_k = c_\infty + A(4.6692)^{-k}$$

Feigenbaum #

CODIMENSION 1 BIFURCATIONS OF PERIODIC ORBITS

A periodic orbit can be codim 1 only if 1 e-value or a c.c. pair of e-values lies on the unit circle.

saddle-node of periodic orbits



period doubling

(dripping faucet: drip – drip – drip – drip – ...

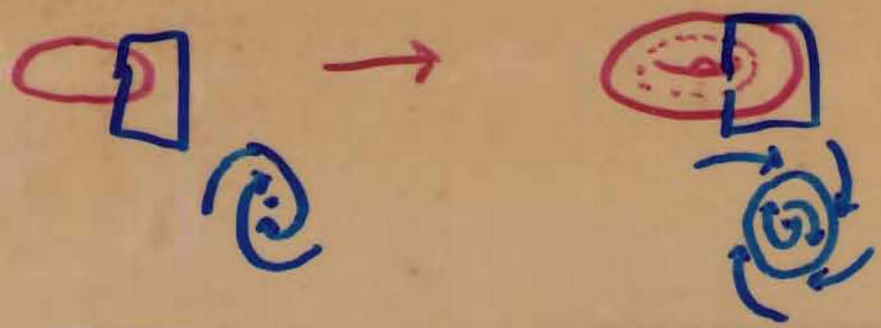
$\Rightarrow$ dripdrip —— dripdrip —— ...)

The resulting orbit itself period doubles ad infinitum and the bifurcation diagram is self-similar.

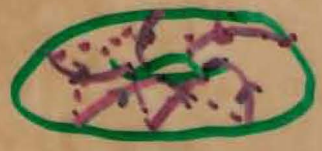
secondary Hopf bifurcation

Arnold shows that this is really codimension 2 since where you leave the unit circle is crucial.

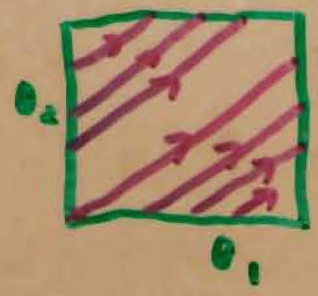
by analogy with v.f.'s expect:



State space for 2 uncoupled limit cycles is a torus:
dynamics is linear $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$



opened up:

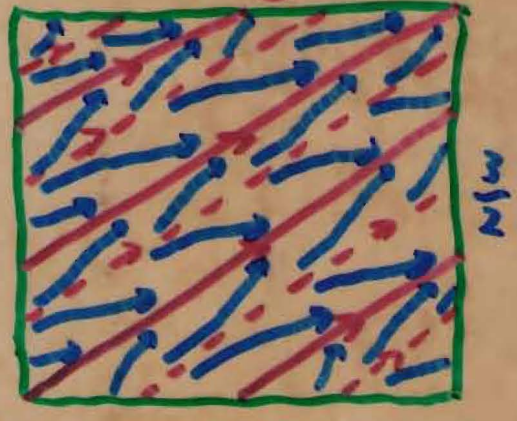


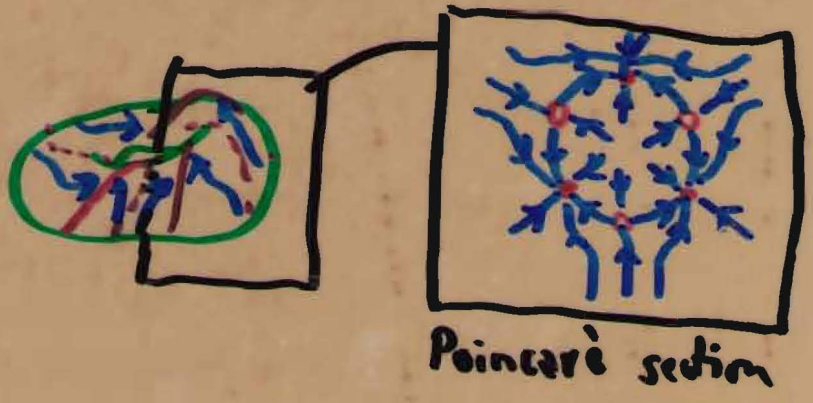
If $\frac{\omega_1}{\omega_2}$ is rational, torus filled with periodic orbits: $\boxed{\text{resonance}}$

If $\frac{\omega_1}{\omega_2}$ is irrational, orbits fill torus densely:

(how quickly it fills up depends on how irrational the ratio is)

Neither of these situations is structurally stable.
A small coupling leads to PHASE LOCKING.

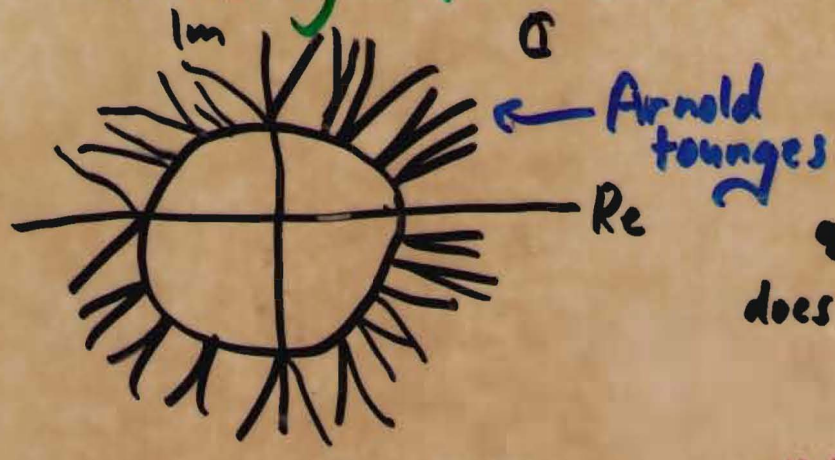




<u>Examples</u>: Pendulum clocks on same wall can synch.
Synchronization of menstrual cycles.

irrationals lock according to their continued fraction expansion.

Secondary Hopf Bifurcation



If eigenvalues leave unit circle at $e^{2\pi i p/q}$ have a resonance of order q.

$q = 1, 2, 3, 4$ = strong resonance does complicated things ($q=4$ not completely understood)

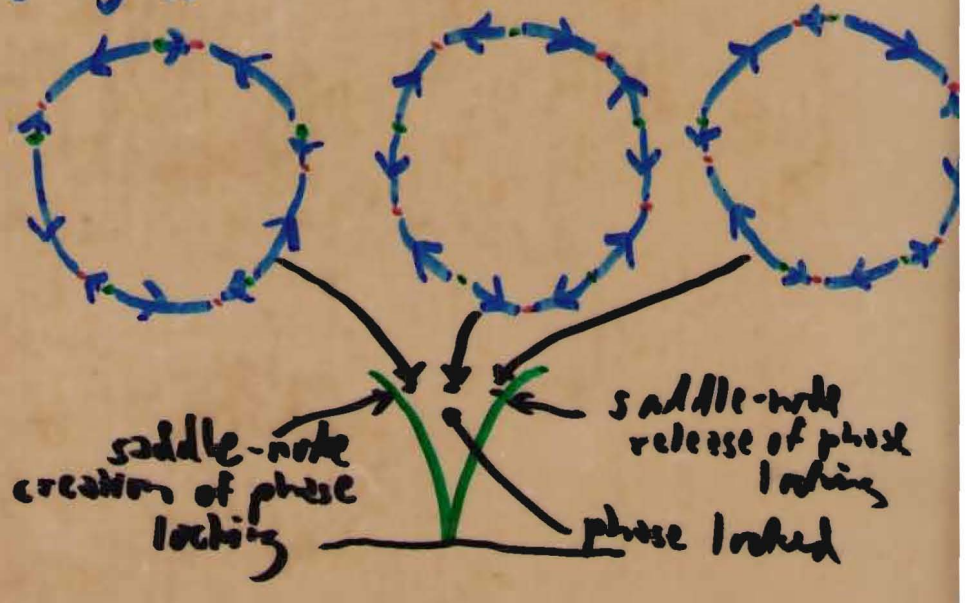
$q \geq 5$ get phase locking in a "tounge" at every rational $\frac{p}{q}$

width $\sim$ (height)$^{(q-2)/2}$

One can remove an interval from around each rational and still have stuff left!

take $\frac{\epsilon}{q^3}$ from $\frac{p}{q}$ - for ϵ small enough total removed length small.)

$\Rightarrow$ close to bifurcation there is positive measure that is not phase locked!



Truncating continued fraction gives best possible rational estimate.

Diophantine numbers have small entries $\Leftrightarrow$ very irrational

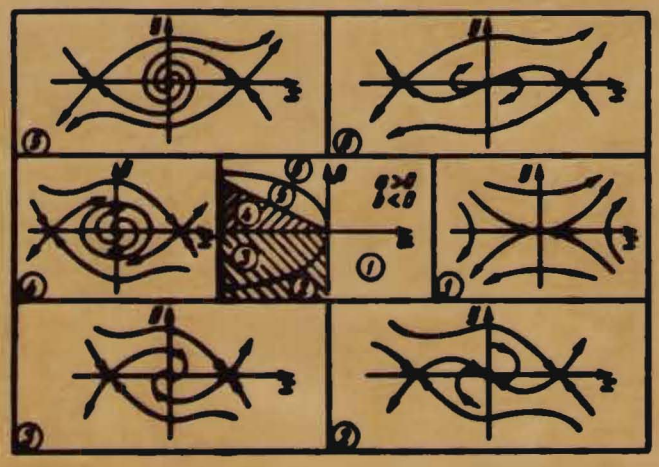
golden mean $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$ is "most irrational"

So get curves with irrational tori between tounges

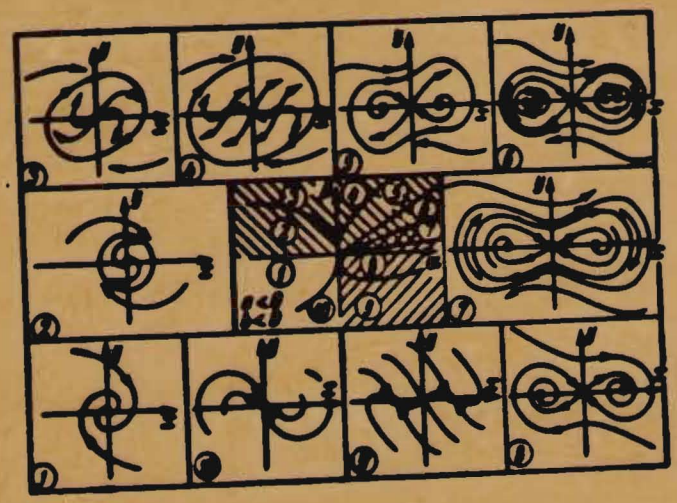
Some irrationals (Louiville numbers) phase lock immediately.

Some become Cantori.

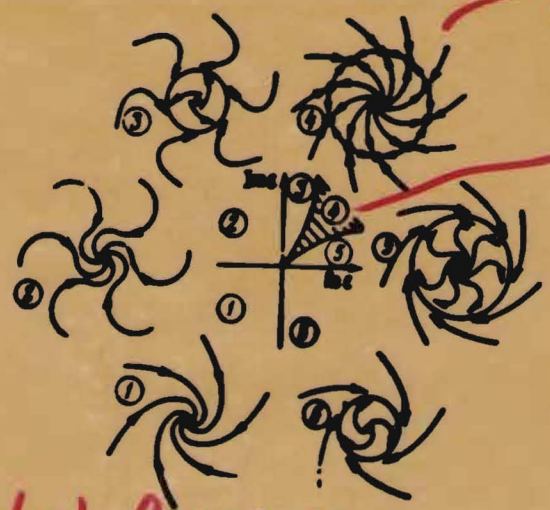
Map - image is smaller gap.... leaves only a Cantor set.



Strong Resonance
Order 2

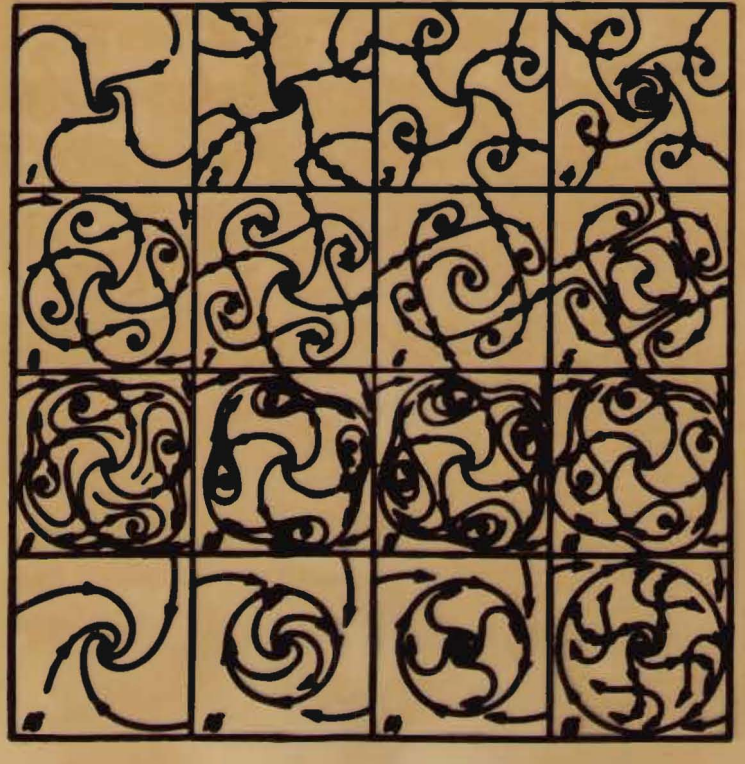


Strong Resonance - Order 3



Weak Resonance
Order 5 and Greater

Strong Resonance - Order 4
(there are others as well)



Classic Cantor Set

recursively remove the middle third of an interval.

equiv: write # in ternary expansion (.0121102 1....)

Cantor set is points with no 1's. Collapse onto an interval by changing $2 \to 1$ and viewing as binary. uncountable # of points

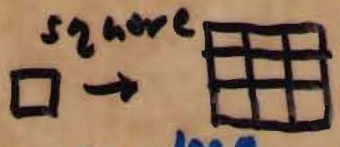
Measure of set is $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

If you blow up set by factor of 3, get 2 copies.

line 2 copies

$1 = \frac{\log 3}{\log 3}$

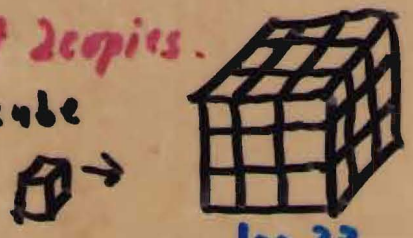
square



9 copies

$2 = \frac{\log 9}{\log 3}$

cube



27 copies

$3 = \frac{\log 27}{\log 3}$

Hausdorff dimension of classic cantor set $= \frac{\log 2}{\log 3} \sim .63$

Fat Cantor Set: Remove middle $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

Get Cantor set with positive measure (dim 1)

(zoom in with microscope, looks denser & denser)

Undernourished Cantor Set: Remove middle $\frac{1}{3}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt[3]{3}}, \ldots$

zero dimensional

(zoom in, gets thinner & thinner)

Mixture: Cut [0,1] at each $\frac{p}{q}$, separate halves by $\frac{1}{q^3}$.

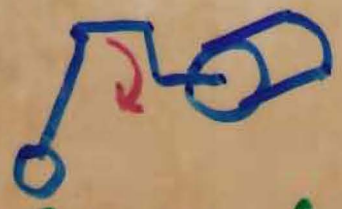
positive measure

zoom in on strong irrationals, looks denser & denser

weak " , looks sparser & sparser

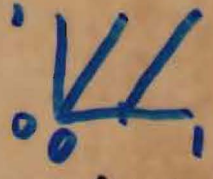
dimension varies wildly

CHAOS

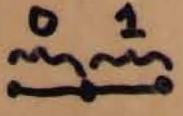


Shaking pendulum: Sequence of clockwise, counter-clockwise
can be any string of 1's & 0's with appropriate initial conditions.
Knowing entire past history of c, cc, c, c, cc, ... tells nothing about next s

Consider the map: $x \to 2x \bmod 1$ of $[0,1]$ to itself.



Or on a circle:



Call left half of interval 0, right half 1.

Claim: There is an initial condition that hops left to right according to any sequence of 0's & 1's.

So this deterministic system produces 0's and 1's as random as flipping a coin.

In binary multiplying by 2 just shifts the binary point.
mod 1 lops off the integer part.
Left half of $[0,1]$ if first digit is 0, right half if 1.
So string of 1's & 0's is digits of binary expansion

.0110001011000 01...

Mapping smooth dynamics to space of symbol sequences is symbolic dynamics.

$2x \bmod 1$ can't be a return map since non-invertible.

Make invertible (but not continuous) in baker's transformation:

$(x,y) \in [0,1]\times[0,1]$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' = 2x \bmod 1 \\ y' = \frac{y}{2} \text{ if } x \in [0,\frac{1}{2}] \\ \quad = \frac{y+1}{2} \text{ if } x \in [\frac{1}{2},1] \end{pmatrix}$

x coordinate does $2x \bmod 1$ but don't lose information since y keeps track.

symbolic dynamics

0	1

write both x and y in binary:

$$\ldots \underbrace{011000101110}_{y} . \underbrace{1011001\ldots}_{x}$$

Map just shifts binary point to the right.

$\Rightarrow$ chaotic orbits of all types

$\Rightarrow$ periodic orbits of every period

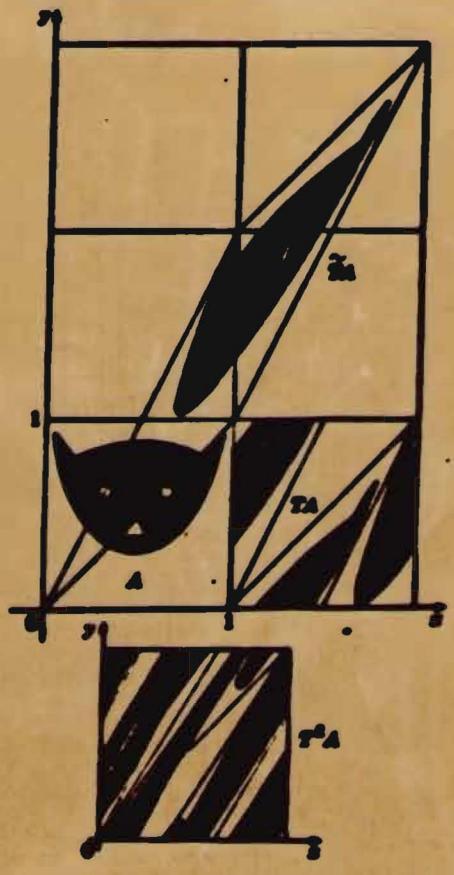
$\Rightarrow$ $\exists$ dense orbit (take $\cdots \underbrace{0}\,\underbrace{1}\,\underbrace{00}\,\underbrace{01}\,\underbrace{10}\,\underbrace{11}\,\underbrace{000}\,\underbrace{001}\,\underbrace{010}\cdots$)

$\Rightarrow$ periodic orbits are dense.

but not continuous.

determinant = 1

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+y \end{pmatrix}$$

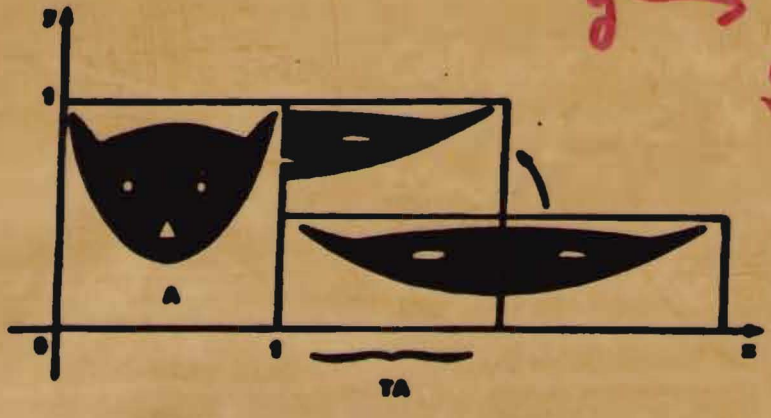


A drawing of the mapping specified by (9). After only two iterations of the mapping, one already has a crazy, mixed-up set.

Arnold's cat map

$$x \longrightarrow 2x \bmod 1$$

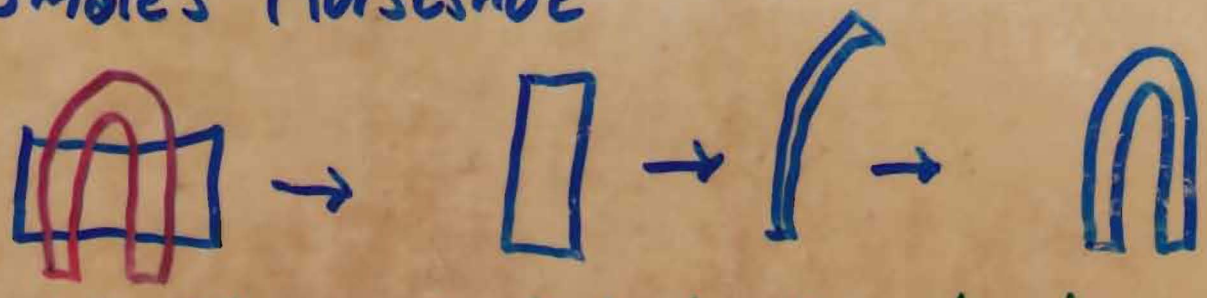
$$y \longrightarrow \begin{cases} \frac{y}{2} & \text{if } x \in [0, \frac{1}{2}) \\ \frac{y+1}{2} & \text{if } x \in [\frac{1}{2}, 1) \end{cases}$$



A drawing of the baker's transformation.

Baker's Transformation

Smale's Horseshoe



Only interested in parts that stay in rectangle
(but could cap: )

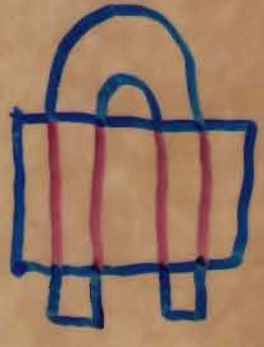
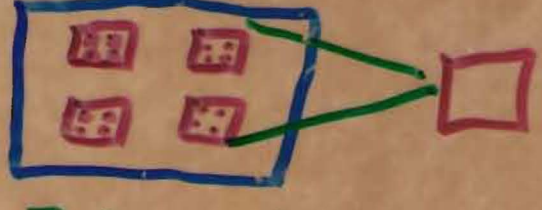


image of red

preimage of red

Things that stay in rectangle forever = Cantor set × Cantor Set

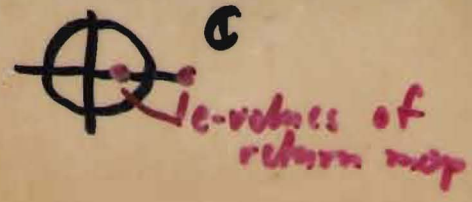


Collapse invariant Cantor set to a square to get baker's transf.

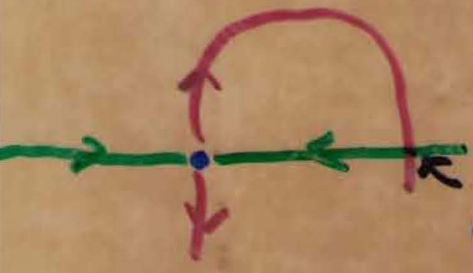
$\Rightarrow \exists$ horseshoe $\Rightarrow \infty$ many periodic orbits of every period, chaotic orbits

$\boxed{\text{horseshoe is structurally stable}}$

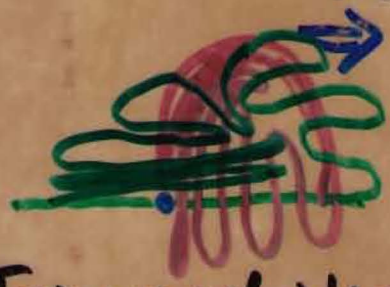
Consider a "saddle" periodic orbit



If stable & unstable manifolds intersect then transverse generically.

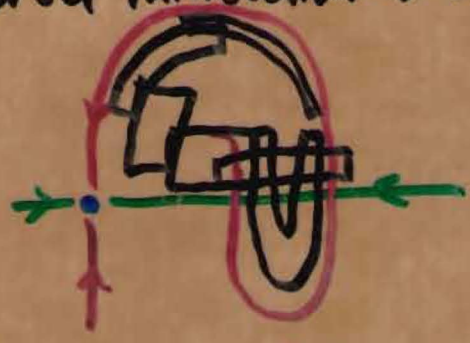


← intersection called a homoclinic pt.
its image & preimage must also be on both stable & unstable manifolds



$\Rightarrow \infty$ # of intersections

Transversal intersection $\Rightarrow$ horseshoe $\Rightarrow$ chaotic orbits

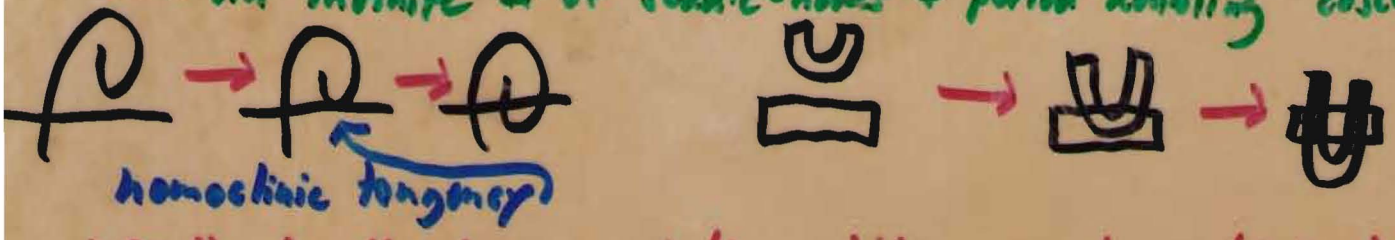


eg: shaking pendulum
restricted 3-body problem
asymmetric top
⋮

SHOEMAKING

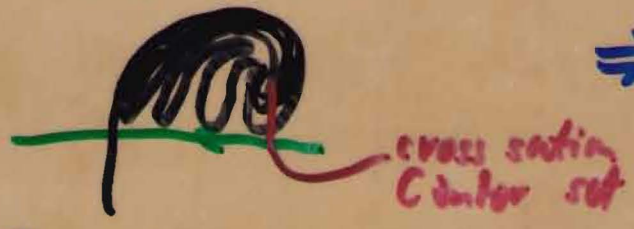
How do the ∞ # of periodic orbits get formed?

→ an infinite # of saddle-nodes & period doubling cascades



→ ∞ # of attracting periodic orbits near homoclinic tangency

Unstable manifold winds around – limit is Cantor set



⇒ 1 homoclinic tangency means an ∞ # nearby

But stable manifold does this too:

⇒ Can get homoclinic tangency when 2 Cantor sets intersect

If they had positive measure and the sum of the measures was greater than the interval

⇒ CONTINUUM OF HOMOCLINIC TANGENCIES

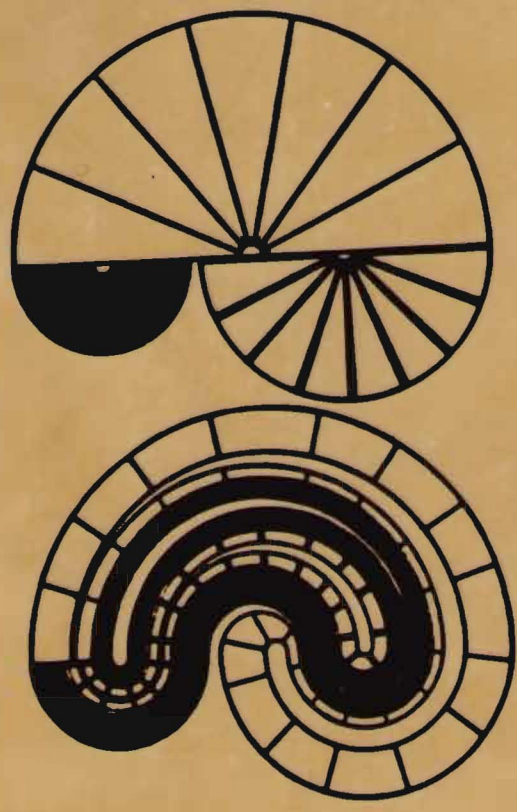
(Really they have measure 0 but can have "thickness which implies the same thing)

Key to Chaos is stretching. Brings noise on the small to the large scale.

ex Fly & the Taffey Puller – Mixing Systems

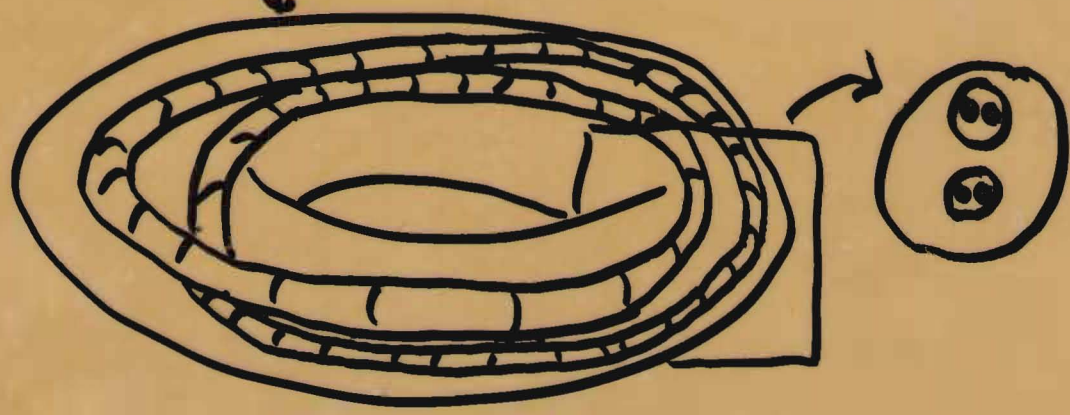
Can make "strange attractors" – fractal dimension.

Plykin
attractor
2-d map
Axiom A

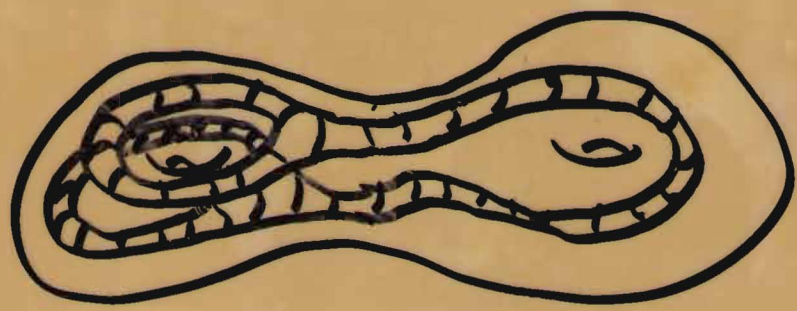


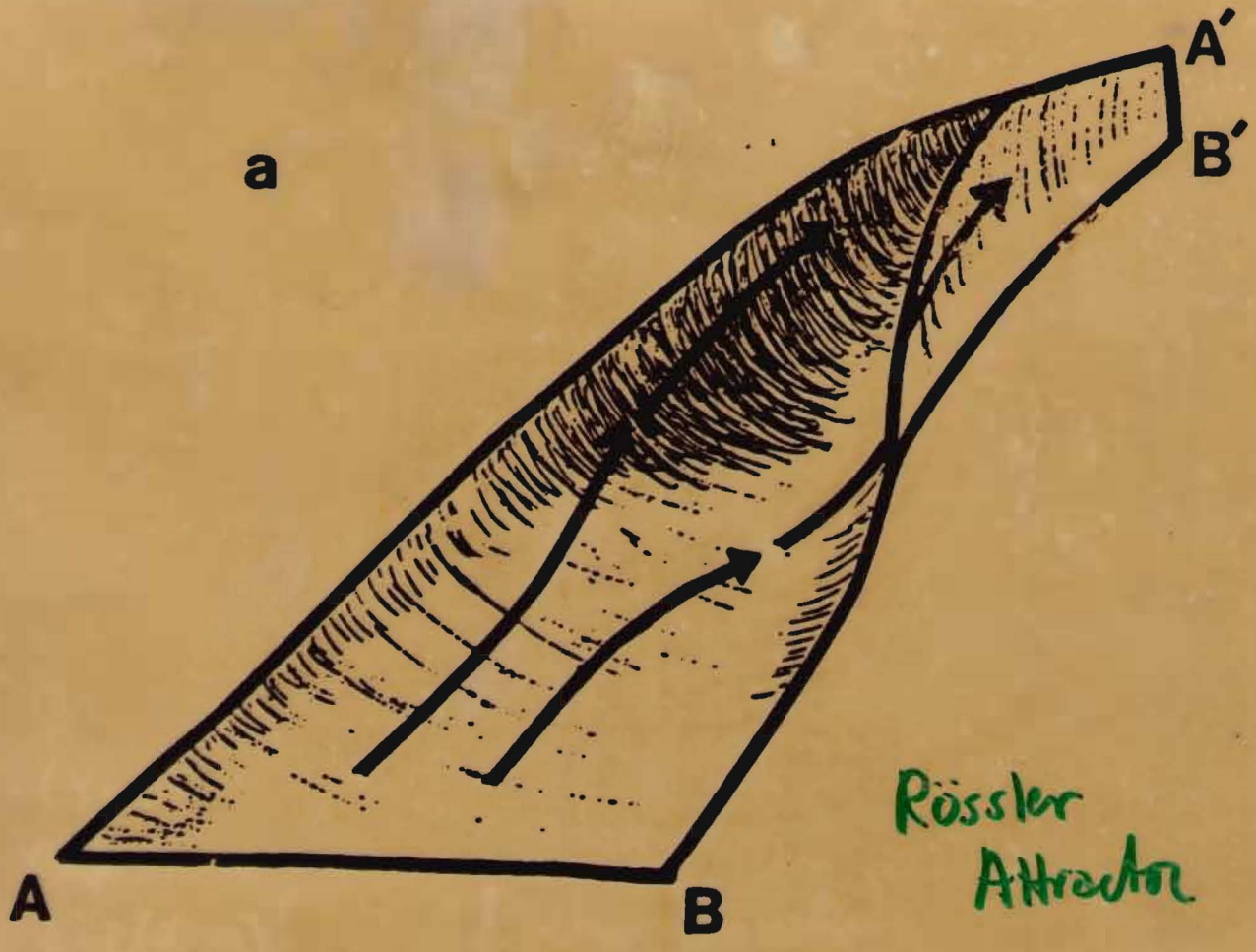
Solid Torus

The
Solenoid

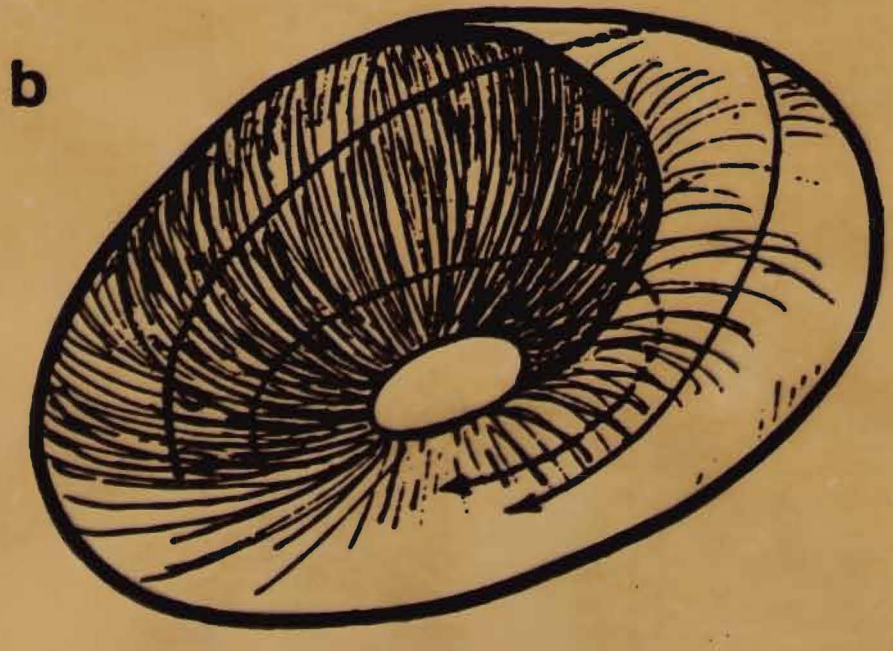


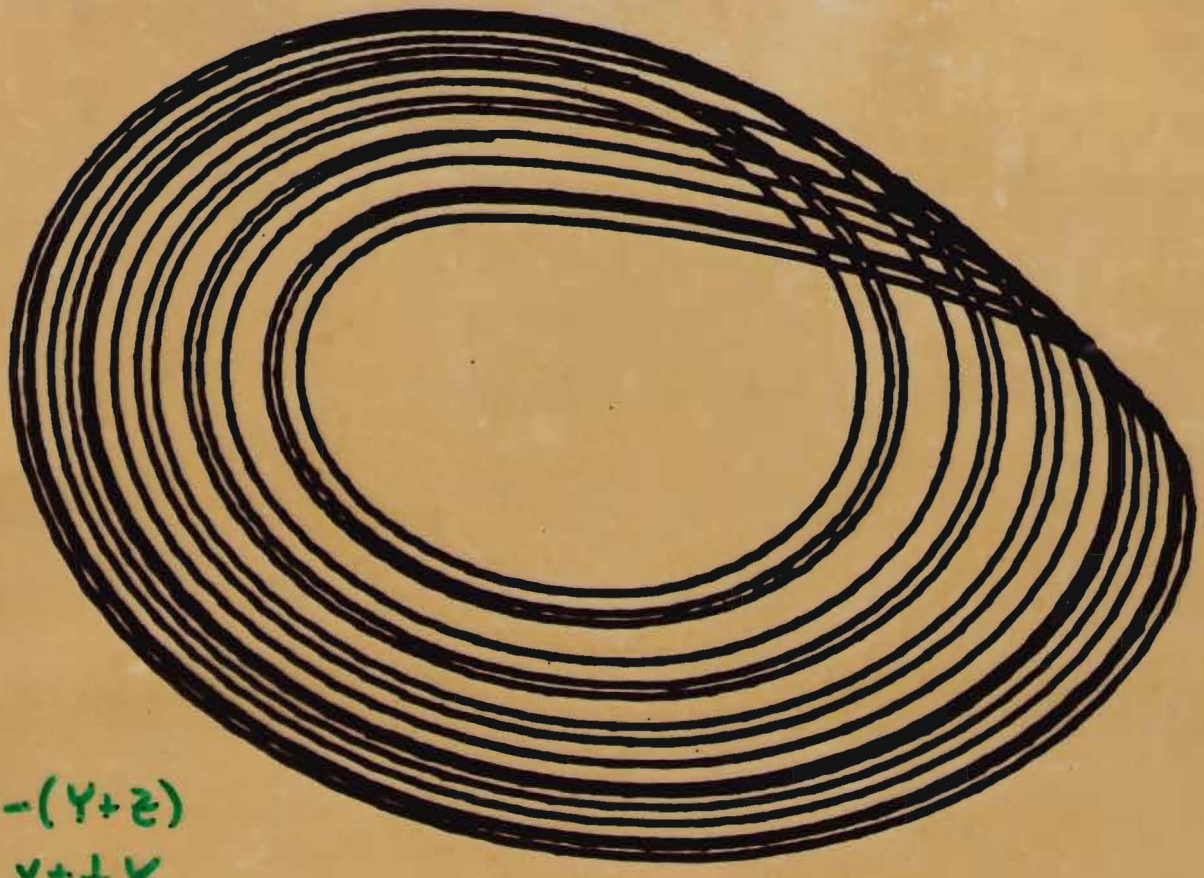
Generalize





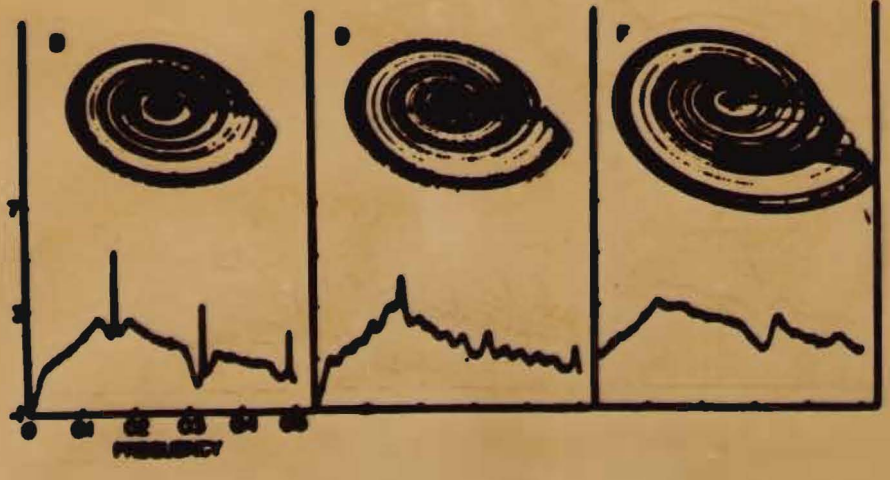
Rössler Attractor

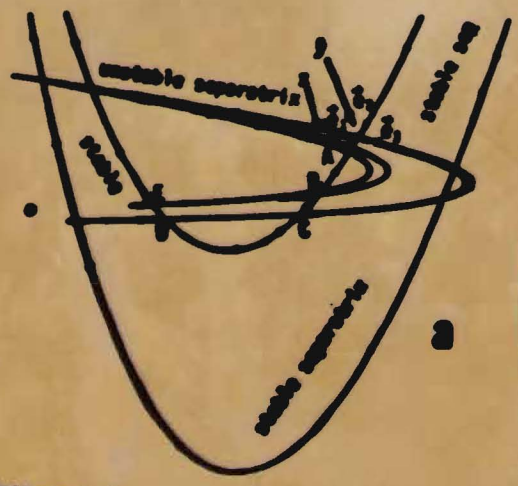




$$\dot{x} = -(y+z)$$
$$\dot{y} = x + \tfrac{1}{5}y$$
$$\dot{z} = \tfrac{1}{5} + z(x-m)$$
$$m = 5.7$$

Rössler Equations

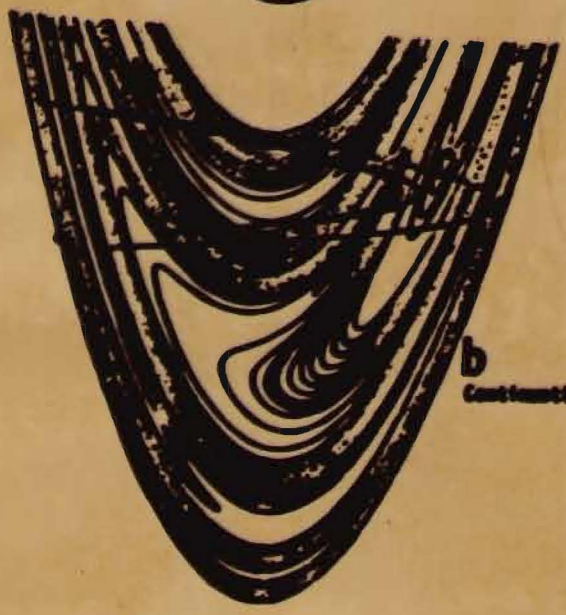


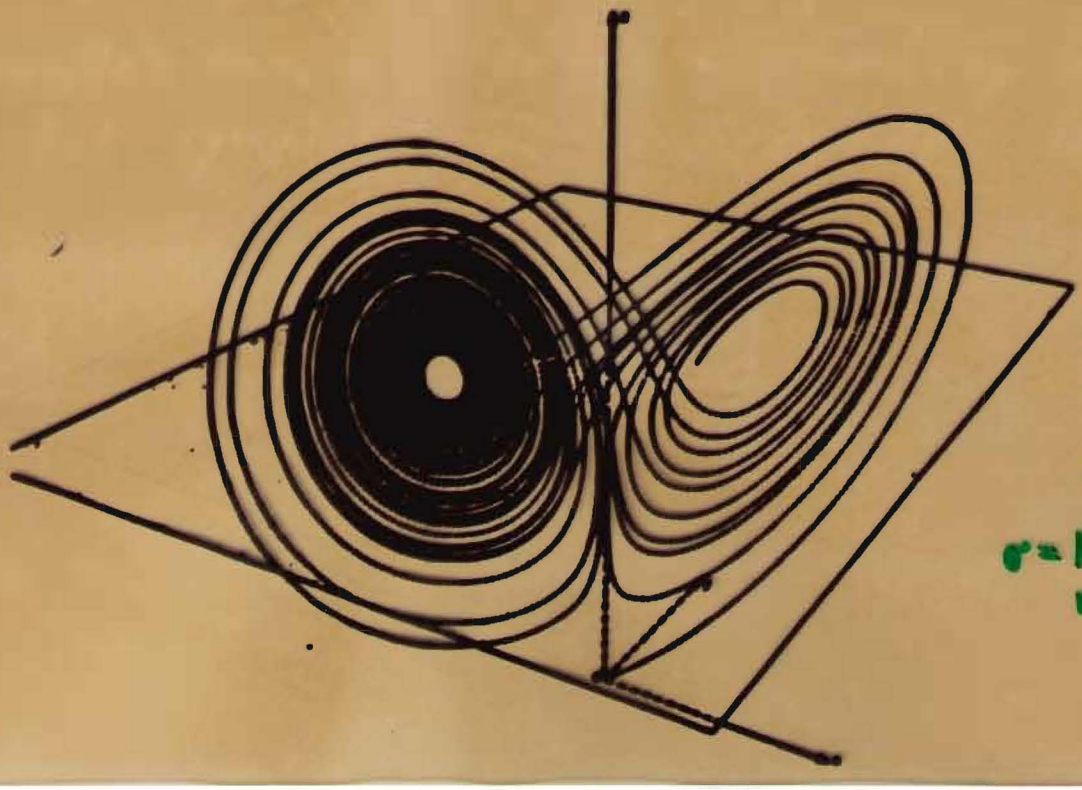


Hénon Attractor

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y+1-ax^2 \\ bx \end{pmatrix}$$

$b=.3 \quad a=1.4$





Lorenz Attractor

$\dot{X} = -\sigma X + \sigma Y$

$\dot{Y} = -XZ + rX - Y$

$\dot{Z} = XY - bZ$

$\sigma = 10 \quad b = 8/3$

$r = 28$

HAMILTONIAN SYSTEMS

Point & Rigid Body Mechanics

Ray equations of Geometric Optics

Fundamental models in every area of physics

basic model: state space is even dimensional - phase space

= configuration space variables q_i $1 \le i \le n$

& their conjugate momenta p^i

Poisson bracket: $\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial q_i} \right)$

Hamiltonian function: $H(q_1, \dots, q_n, p_1, \dots, p_n)$

Dynamics: $\dot{f} = \{f, H\}$

Special Properties:

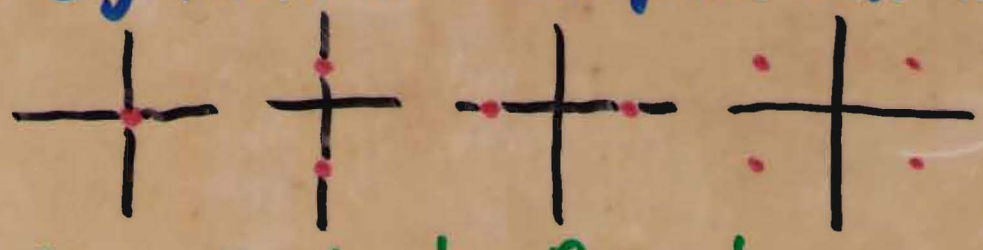
Noether's theorem: symmetries $\leftrightarrow$ conserved quantities

$0 = X_J \cdot H = \{H, J\} = -\{J, H\} = -\dot{J}$

Liouville's theorem: volume preserved by flow

in fact symplectic form $\omega = \sum_i dq_i \wedge dp^i$ is

$\Rightarrow$ Eigenvalues at fixed points must look like:



No attractors! Poincaré recurrence theorem

Only solvable systems were ones with lots of symmetry.

Noether → constants of motion in involution.

If can find n constants of motion I_i s.t. $\{I_i, I_j\} = 0$ called INTEGRABLE. ⇒ can find action angle variables

⇒ Dynamics is on n-dim tori.

Nobody could deal w/ anything else - eg. 3-body problem

Issues clear even with 2-coupled oscillators.

Phase space is 4-dim, H cuts down to 3-dim, filled with 2-dim tori, take Poincare section.

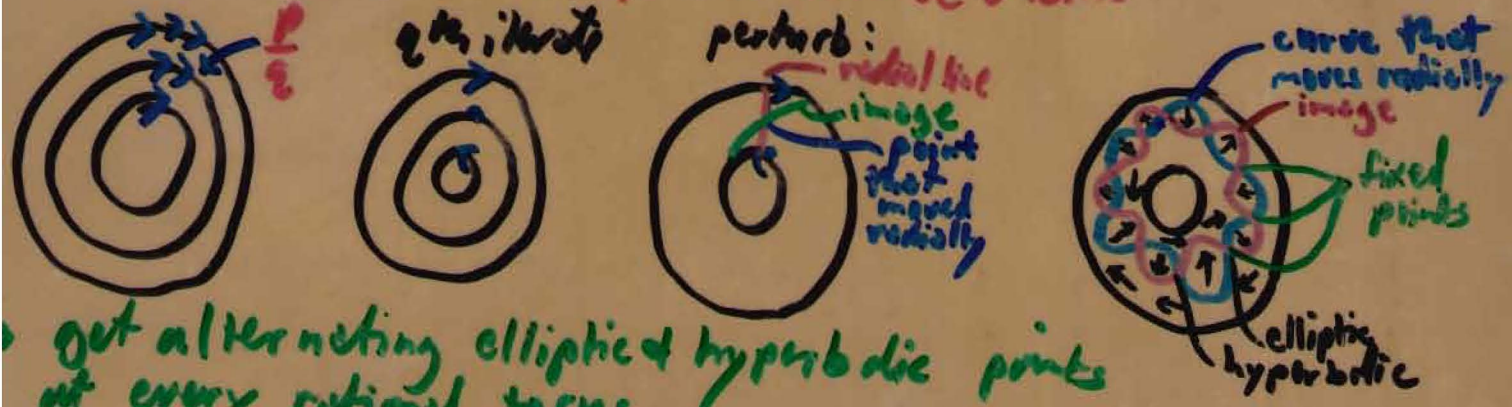
Get elliptic  & hyperbolic  fixed points

(Must be area-preserving)

KAM theorem - for small enough perturbation of integrable system, positive measure of tori persist.

Numerically, frequency varies as the action does.

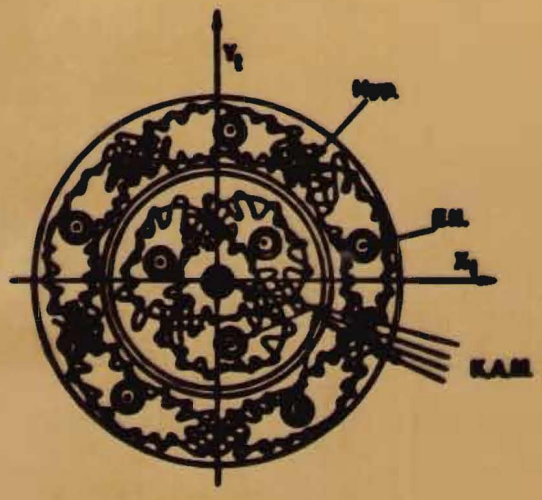
⇒ Rational tori break up into island chains

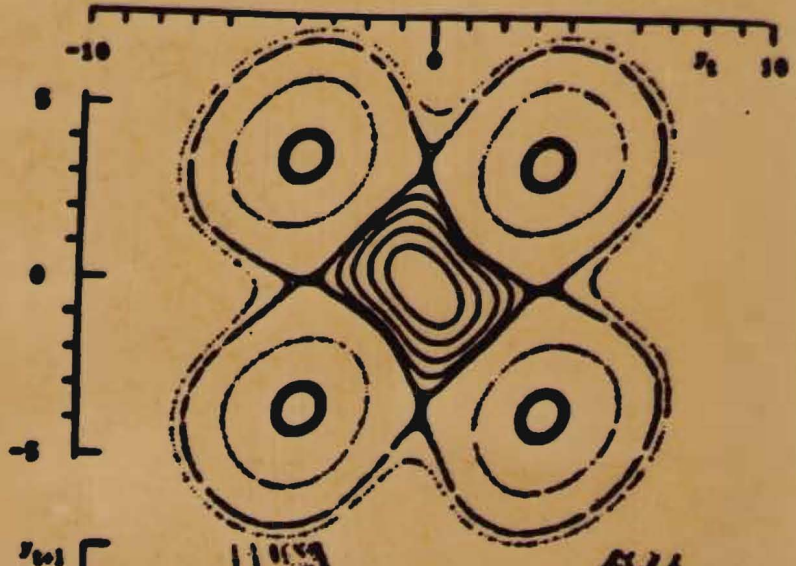


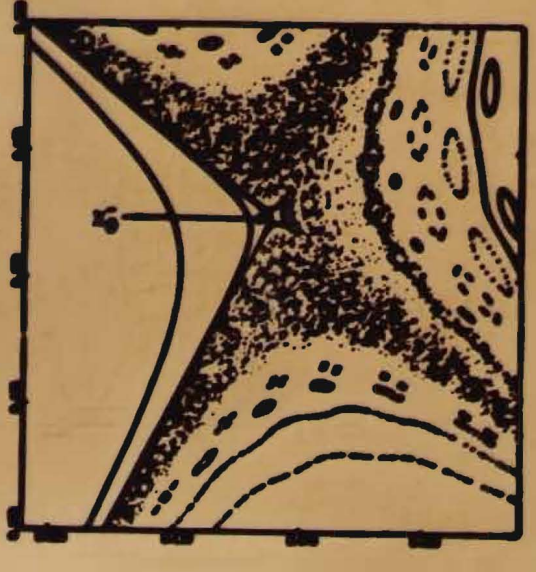
get alternating elliptic & hyperbolic points at every rational torus.

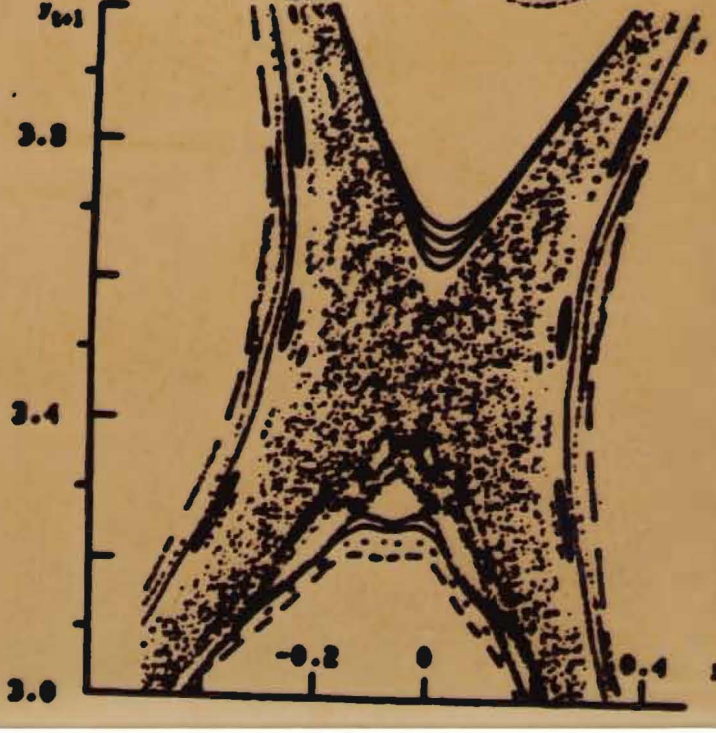
The hyperbolic pts have homoclinic behavior - horseshoes, etc.

The elliptic points are surrounded by tori that are themselves breaking - so whole picture repeated ad infinitum.


KAM

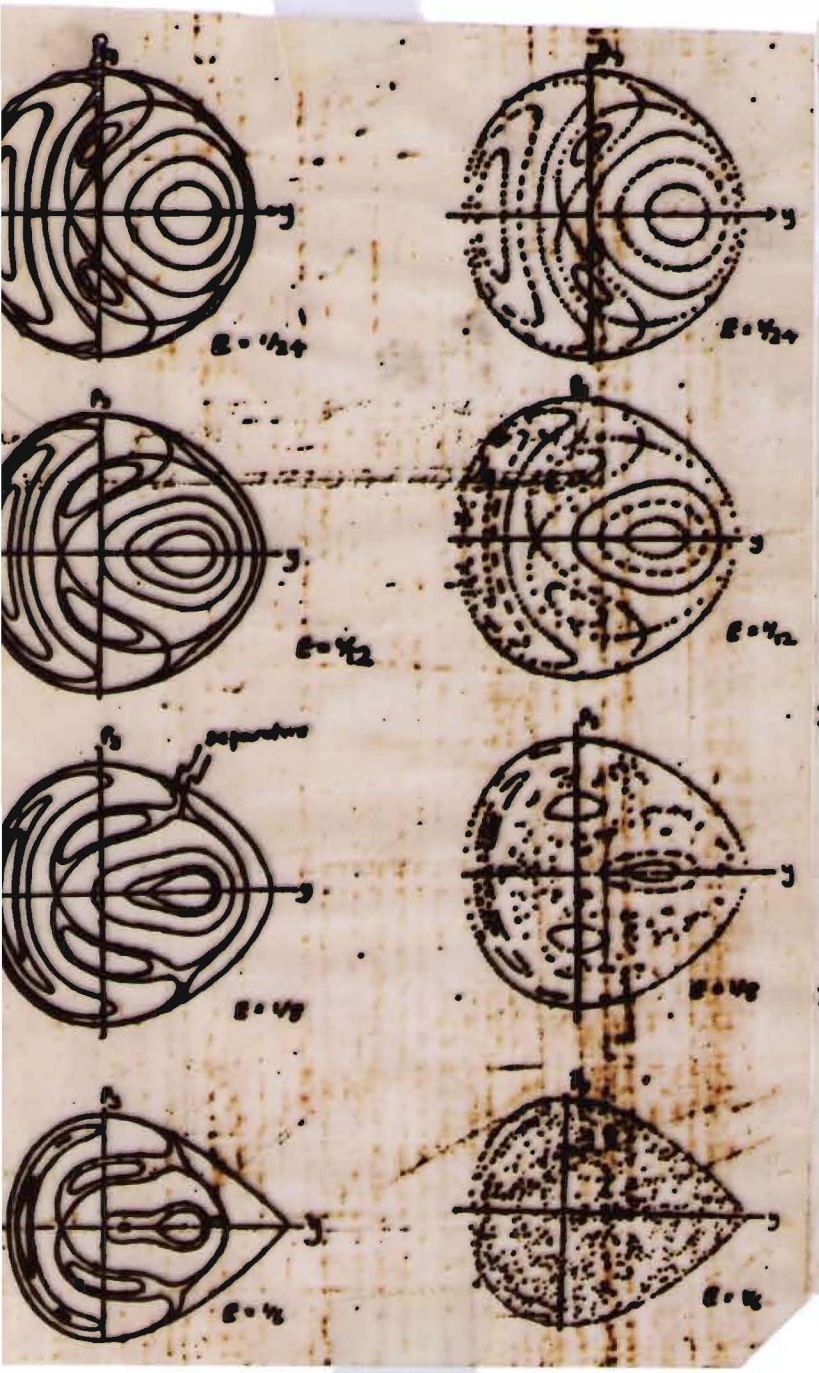

-10
10
5
0
-5




3.8
3.4
3.0
-0.2
0
0.4

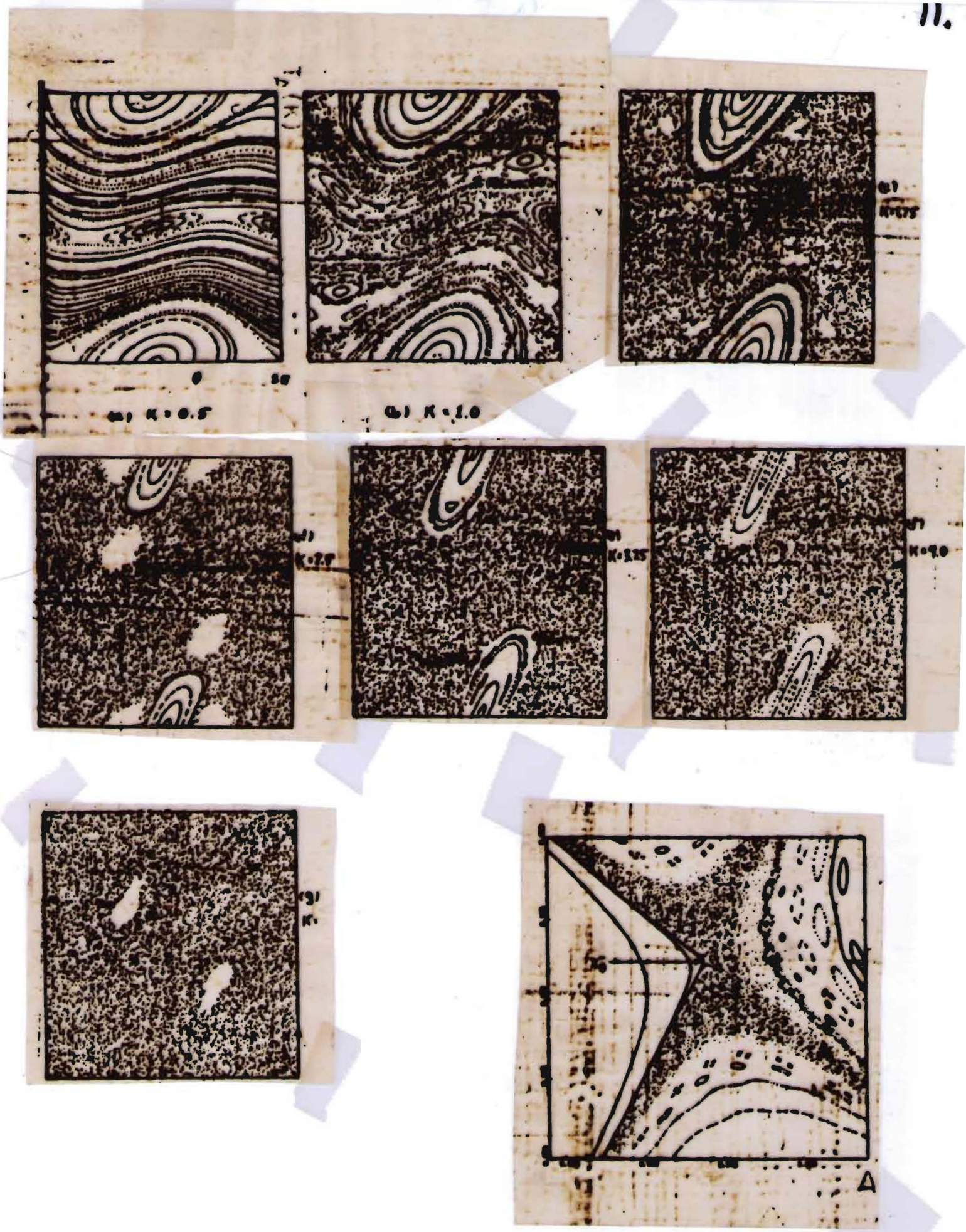
Some orbits of the standard map at k = 0.9

Note the intricate self-similar, "fractal" structure, where islands whose structure is a small copy of the whole, form in between separatrices which form "horseshoes" and are tems chaotic.



Hénon-Heiles system - surface of section: $V(x,y) = \frac{1}{2}(x^2+y^2)+x^2y-y^3/3$
Model for star motion in a galaxy
Left is perturbative approximation to a "const. of motion".

Close-up of chaotic regim near separatrix (dots are a single orbit)


0
2π
ω1 K = 0.5
ω1 K = 1.0
ω1
K = 0.75
ω1
K = 2.5
ω1
K = 2.25
ω1
K = 9.0
ω1
K =
Δ

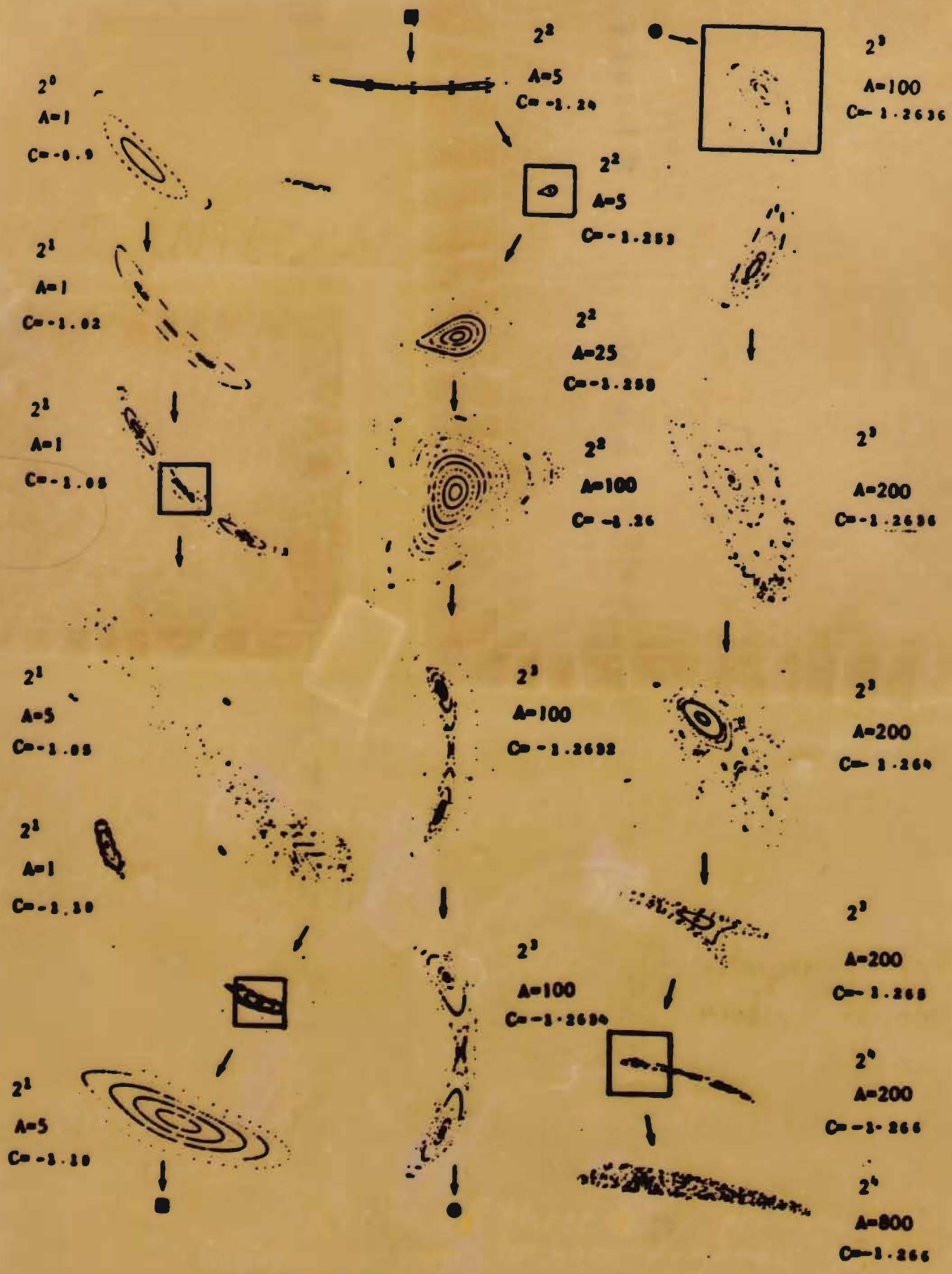
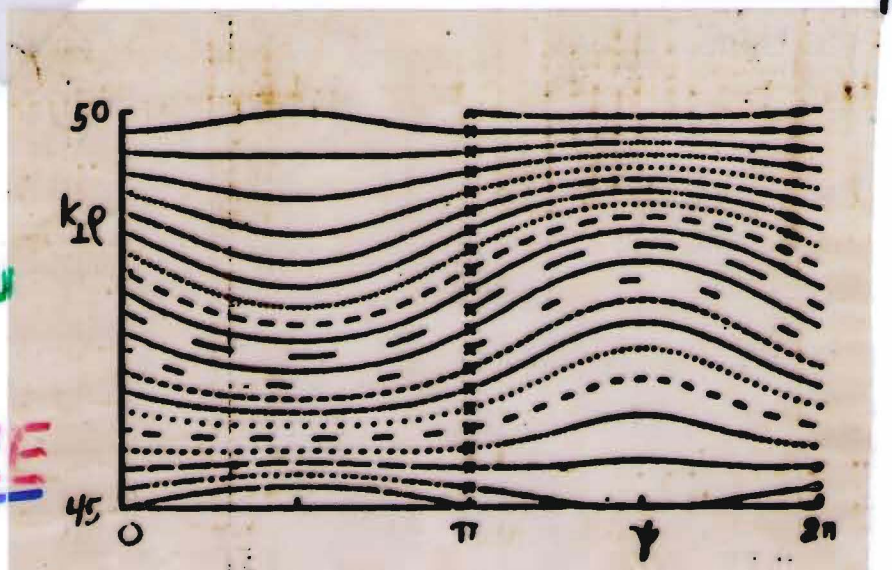
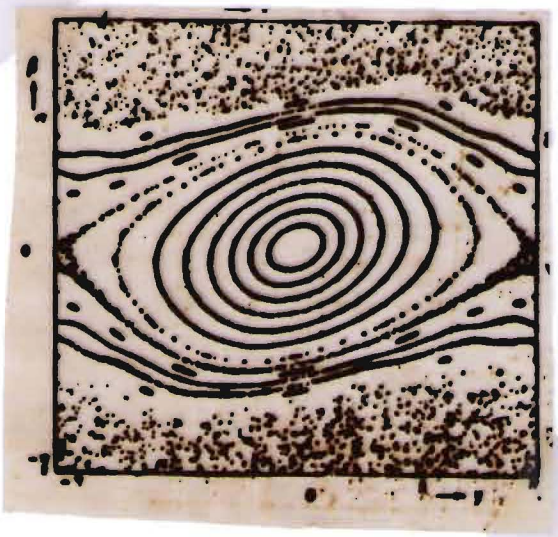


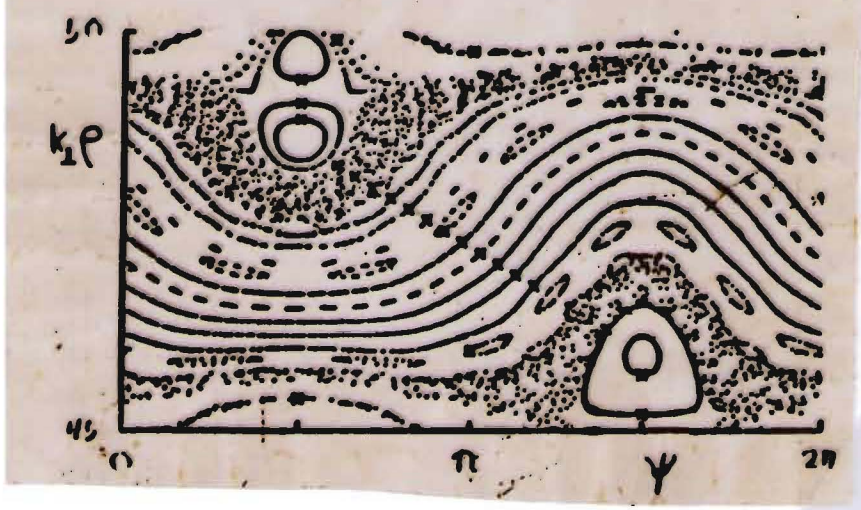
Figure 2. Phase plots of the period-doubling conservative Feigenbaum [Eqs. (4.1)–(4.9)],[7] that is, x_i horizontally versus x_{i-1} vertically. A sequence of magnifications A of the regions about the orbits of period 2^k, starts at the top, on the left, going downward. The C is the parameter in (2.6).

When the phase space is this mixture of tori and chaotic regions the system is called

QUASI-INTEGRABLE

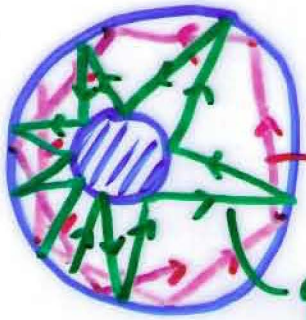






A theorem for billiards similar to K.A.M. holds only if boundary has 533 derivatives.

eg The Kitaki billiard:



whispering gallery modes – regular

quasi-integrable region

The three body problem is quasi-integrable in most regimes.

heavy equal masses

light test mass

Restricted three body problem has been shown to have dynamics governed by any page torn from any book of random numbers for some initial condition.

p= -1.000000 x= -0.500000 to -0.000000 y= -0.005000 to 0.005000

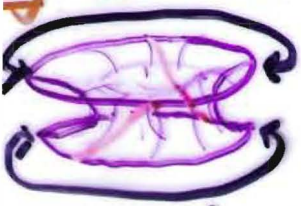
Poincaré de sistema linear

p= -0.4224000 xmn= -0.87 xmx= -0.73 ymn= -0.80 ymx= -0.45 n= 10000

When perturbed far enough we find full-fledged chaos.

Orbits diverge exponentially from one another ⇒ arbitrary changes in initial conditions (eg. gravitational effect of an electron on Jupiter) can completely change subsequent evolution. Thus knowing the system's history tells us nothing about its future, though deterministic, it appears random. Eg. The Stadium billiard: 

Eg. Geodesic flow on negatively curved manifolds – Anasov.

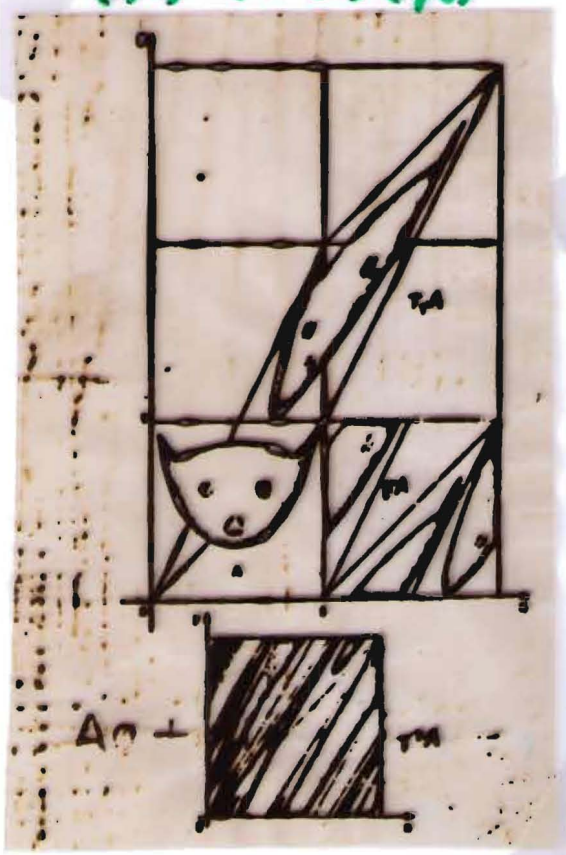


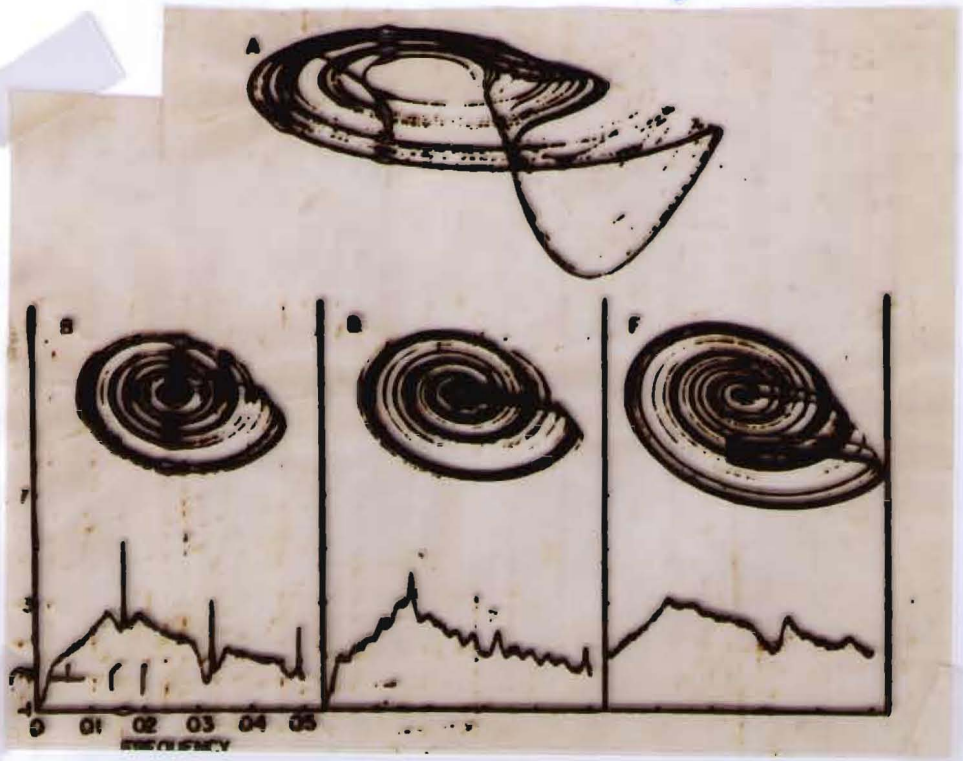
In the limit this becomes toral billiards with round post:  i.e. a hard-sphere gas – shown to be ergodic by Sinai.

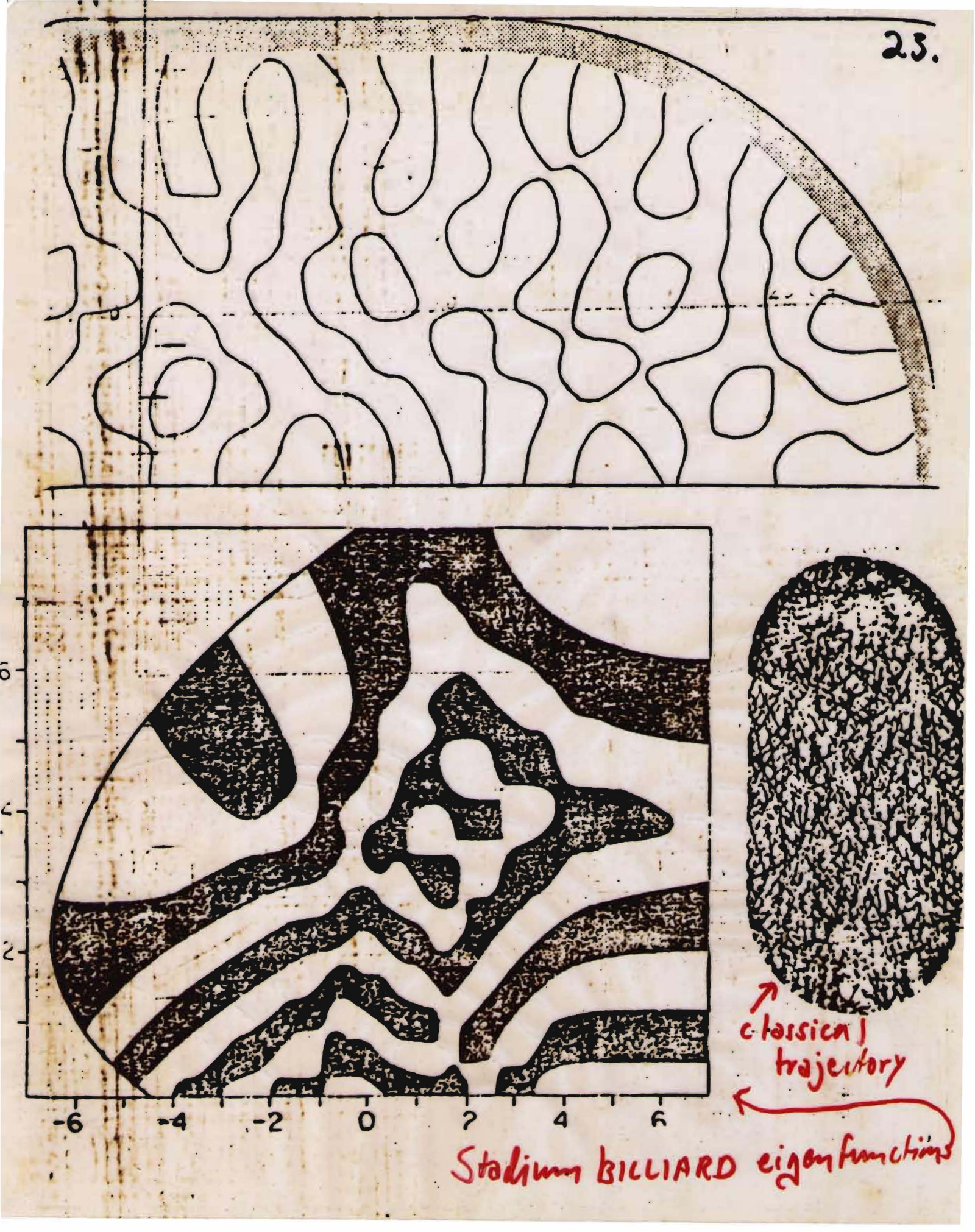
Arnold's "cat map" (can "suspend" to give a flow)

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

In dissipative systems find "strange attractors"







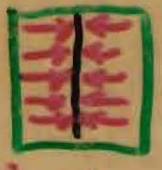
Stadium BILLIARD eigenfunctions

RENORMALIZATION

Sometimes under the appropriate microscope, things look universal.
Define a renormalization operator on the space of all systems.
study fixed pts, basins of attraction etc.

Eg. **Derivative** Consider the functions which vanish at zero.

$Rf(x) = 2f(\frac{x}{2})$ Only fixed pts are linear fns.
They are attracting.



Your derivative is which linear function you are attracted to.

Eg. Saddle node

$R\begin{pmatrix} v_x \\ v_y \end{pmatrix}(x,y) = \begin{pmatrix} 4v_x \\ 2v_y \end{pmatrix}\left(\frac{x}{4}, \frac{y}{2}\right)$

$\dot{x} = x^2 \quad \dot{y} = -y$
codim 1 attracting
1-dim repelling

Every curve which intersects W^s maps onto W^u with enough R iterations.

So W^u is universal way to annihilate a saddle & node

Eg. Feigenbaum Hump maps  f of 

period doubling

$Rf(x) = \alpha f \circ f(\beta x)$

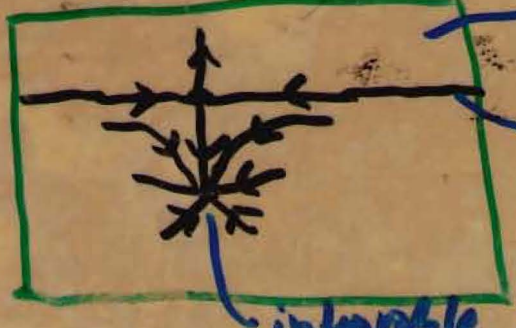
1 dim stable
codim 1 unstable
3rd.
2nd
1st period doubling

In space of hump maps, period doubling surfaces accumulate on W^u.
spacing is R's unstable eigenvalue = Feigenbaum's number.

Eg. KAM and breaking of golden torus

Similar renormalization around golden mean torus yields attracting fixed pt. (KAM torus) and saddle with 1-d W^u.
Universal way to break the last KAM torus.

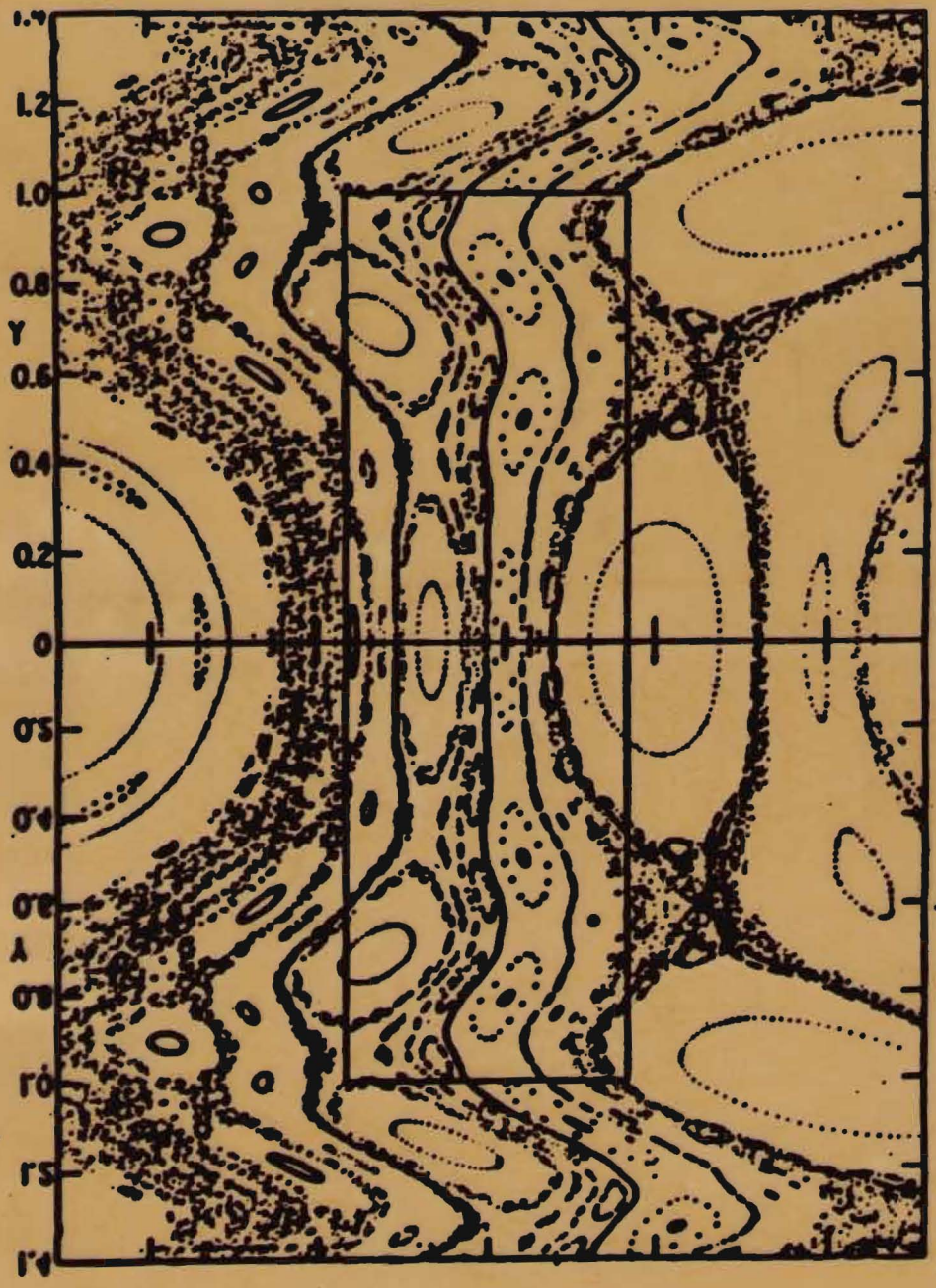
Greene & Mackay



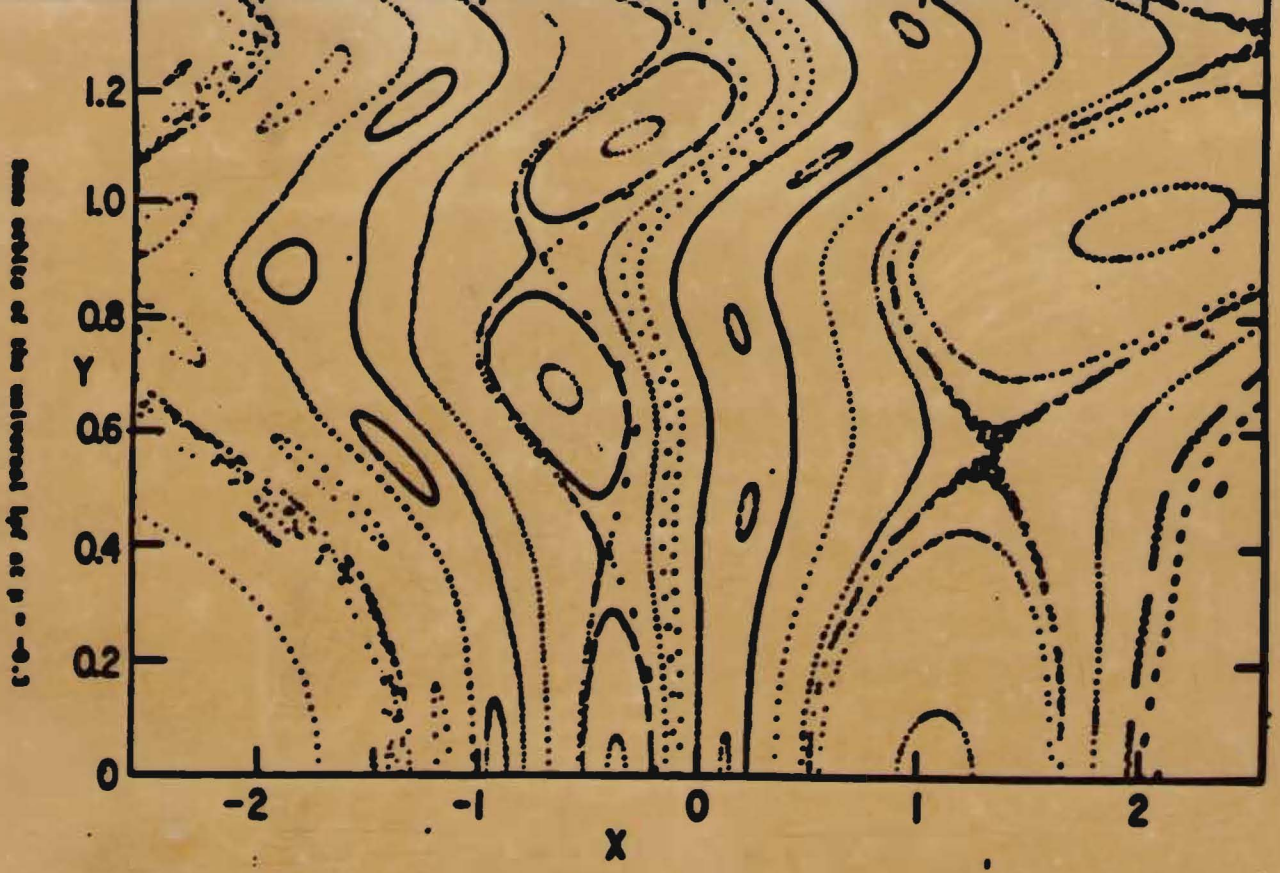
no tori

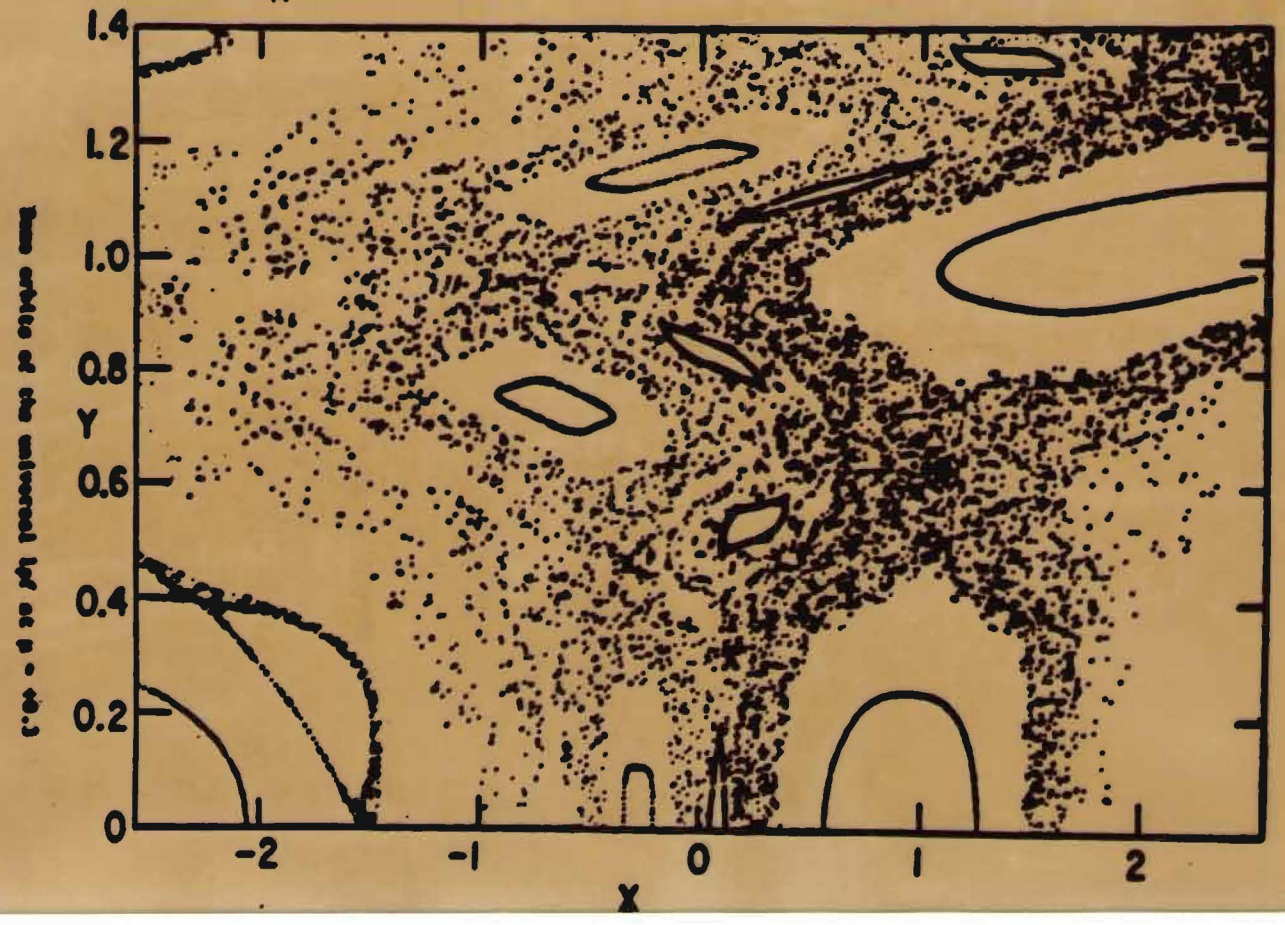
only golden mean left

integrable



Some orbits of the universal map for the neighbourhood of critical orbits circles


1.2
1.0
0.8
Y
0.6
0.4
0.2
0
-2
-1
0
1
2
X

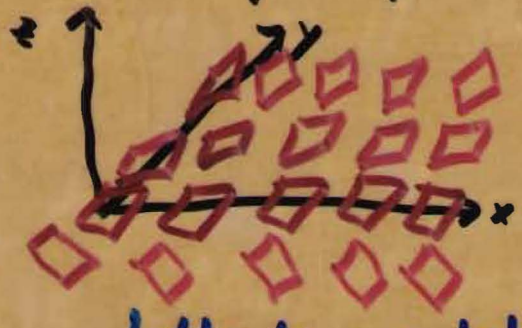

1.4
1.2
1.0
0.8
Y
0.6
0.4
0.2
0
-2
-1
0
1
2
X

Non-integrable distributions

A smooth choice of vector at each point in a space is always tangent to a smooth family of curves

There does not always exist a smooth set of surfaces tangent to a field of 2-dim (or higher) planes:

eg. in 3-dim the 2 space spanned by $\hat{u}_y$ and $\hat{u}_x + y\hat{u}_z$



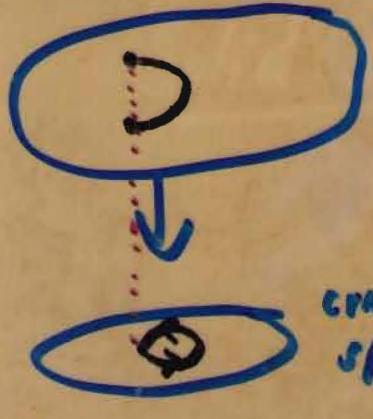
move in little loops
can move ⊥ to planes

This phenomenon leads to non-holonomic constraints
- only the direction you can move in phase space is constrained not what point you are at. (parallel parking)

Lie bracket & exterior derivative measure non-integrability.

Closely related to curvature in fiber bundles

parallel transport in a loop - don't end up where you started



Force a crab to move the tip of a claw in a circle
- leg doesn't return.

If model leg motion decision locally get holonomy - curvature in the "fiber bundle"

SIMPLIFICATION DUE TO SEPARATION OF TIME SCALES -

Approximately:

1) Fast degrees of freedom feel the slow degrees as if they were constant.

2) Slow degrees of freedom feel only the average effect of the fast degrees.

eg: Boltzmann's eqn. from BBGKY - correlation fns.
Quasilinear theory - waves & background
Molecular physics — Nuclei and electrons
Solid state physics —

Dynamical System with parameter λ:

$$\dot{x} = X(x, \lambda)$$

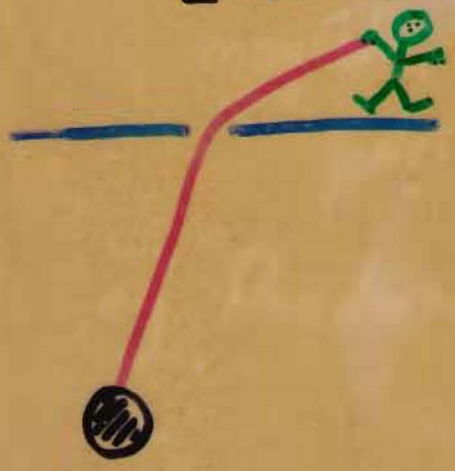
An <u>adiabatic invariant</u> $I(x, \lambda)$ is a function of the state x and parameter λ whose variation over times of order $\boxed{\frac{1}{\varepsilon}}$ is of order ε on solution curves of:

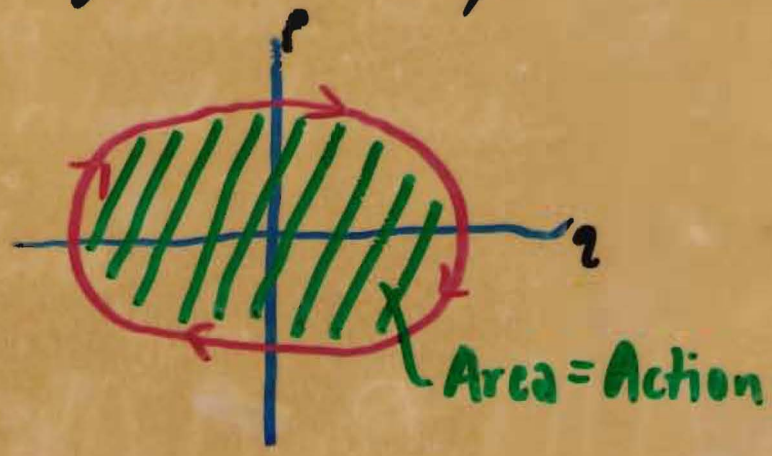
$$\dot{x}(t) = X(x(t), \lambda(\varepsilon t))$$

where $\lambda(\ )$ is a given smooth function.

Example: SHO with slowly varying parameter

Einstein & the old quantum theory





Adiabatic invariant is the action J.

$H = \omega J$ $\qquad$ $H \sim \frac{1}{\sqrt{L}} \Rightarrow \text{force} \sim \frac{1}{L^{3/2}}$

eg: Tuning a Guitar while a note is playing

Feel an extra force! - a pseudoforce

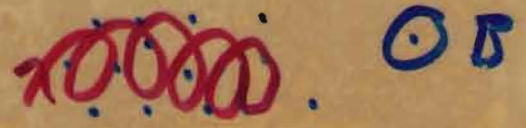
freq of fundamental $\sim \sqrt{\text{tension}}$

so $\frac{H}{\omega} \sim \frac{H}{\sqrt{T}} = \text{const} \Rightarrow$ potential of $\sqrt{T}$

force of $\frac{1}{\sqrt{T}}$

eg: Singing in shower as tub is filling

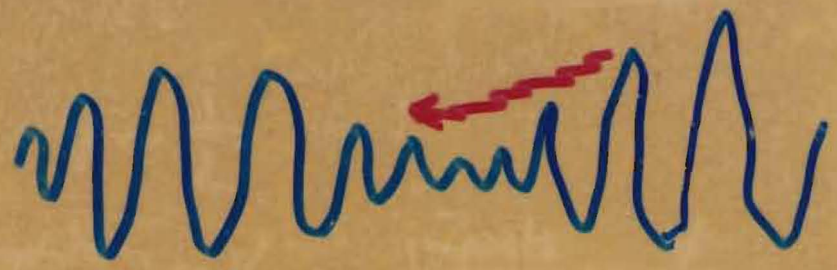
eg. Gyromotion:



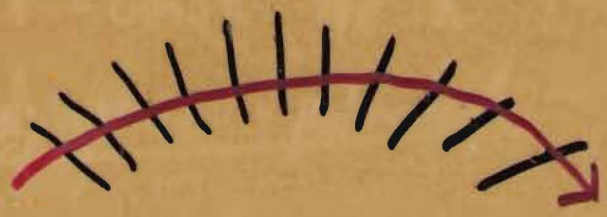
magnetic moment is adiabatic inv.

drifts in reduced dynamics

eg. Pondermotive force:

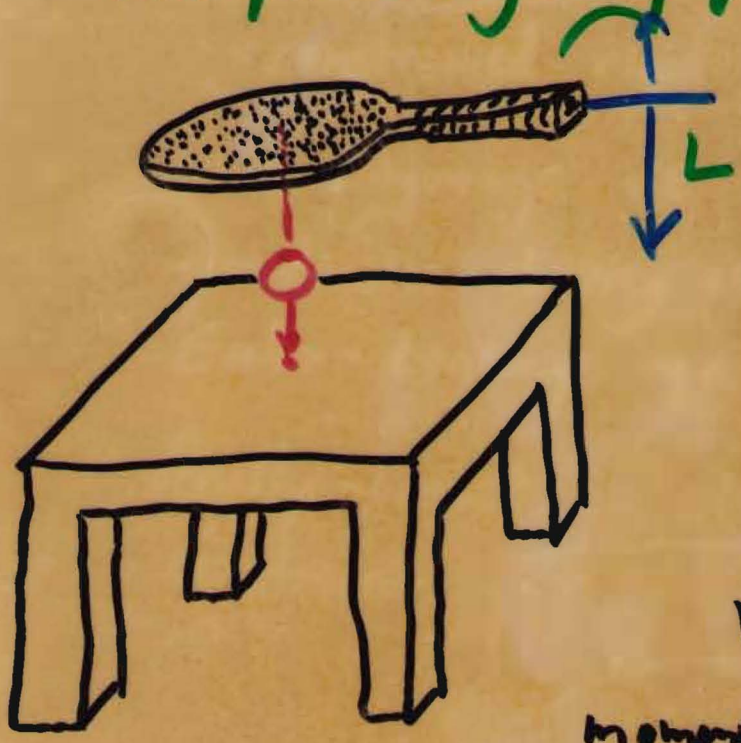


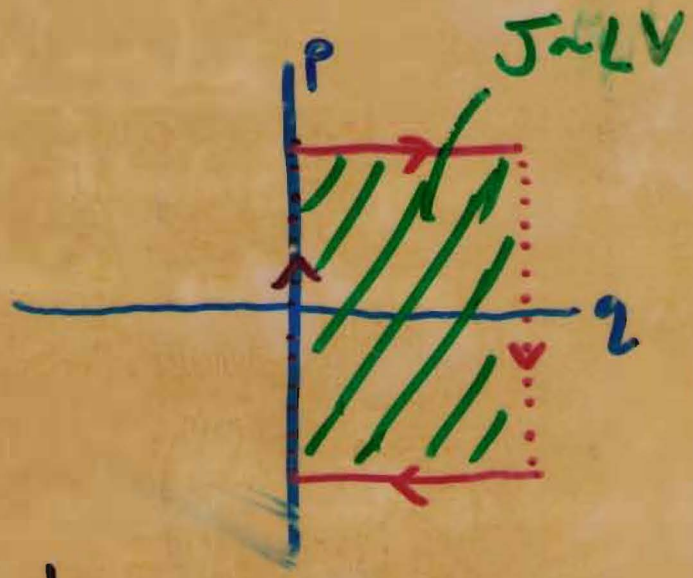
eg. WKB waves: (slowly varying medium)



Relation between ponderomotive force of wave on medium and dielectric "force" of medium on wave.

Example: Ping Pong paddle & ball





$$V \sim \frac{1}{L}$$

$$\frac{\text{momentum}}{\text{impact}} \sim V \qquad \frac{\text{impacts}}{\text{time}} \sim \frac{V}{L}$$

So force $\sim \frac{V^2}{L} \sim \frac{1}{L^3}$ potential $\sim \frac{1}{L^2}$

This is $pV^\gamma = \text{const}$ for a 1-dim gas.

Thermodynamic potential. — pseudoforces

In a formal sense the entropy is an adiabatic invariant for systems ergodic on energy surface.

Late 1800's — All potential energy is really kinetic energy of hidden degrees of freedom

van Kampen –

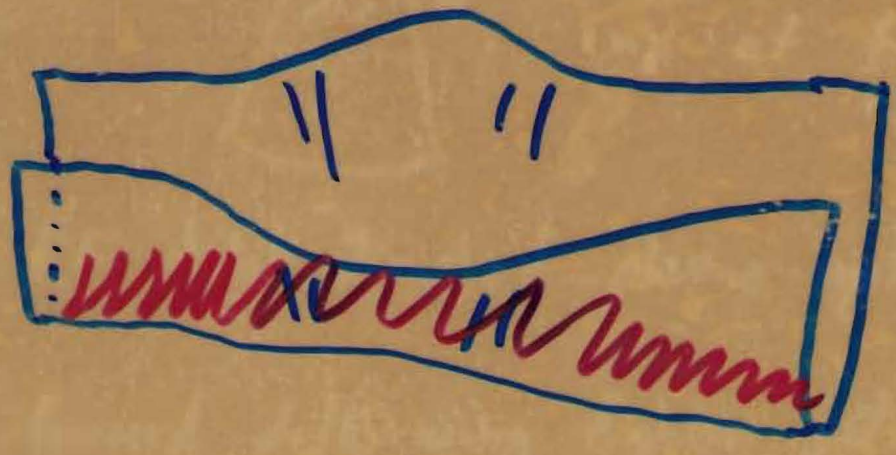
constrained rigid body mechanics

since Lagrange we just introduce generalized coordinates obeying constraints and solve Euler-Lagrange eqn.

Really - must be steep potential well enforcing constraint :

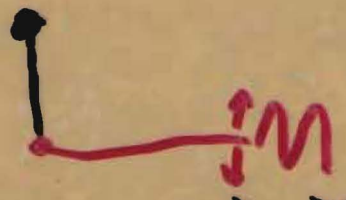
system oscillates rapidly in direction of constraint.

If constraint well width varies, get extra pondermotive forces in constrained motion!

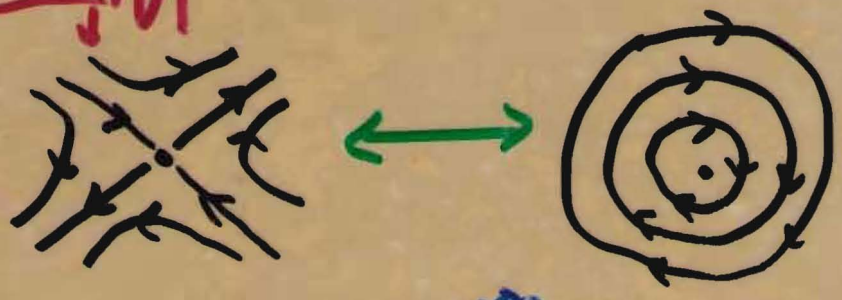


Pseudoforces can stabilize unstable fixed points.

eg. shaking inverted pendulum:

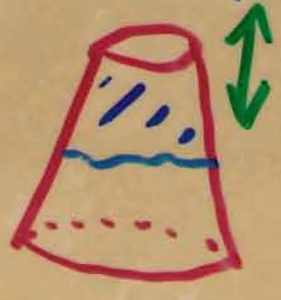


in phase space:



if fast enough, average is:

eg. Can stabilize Rayleigh-Taylor!



Shake an inverted cup of fluid

eg. RF stabilization of plasma instabilities

eg. forward delta wing fighter jets

aerodynamically unstable
stabilized by fast computer